CHAPTER 1: MATRICES AND DETERMINANTS

2 - MARKS, 3 -MARKS (5- MARKS ONLY QUESTIONS GIVEN)

2 MARKS

EXERCISE 1.1 : 1(i). Find the adjoint of $\begin{pmatrix} -3 & 4 \\ 6 & 2 \end{pmatrix}$

Solution:

$A = \begin{pmatrix} -3 & 4 \\ 6 & 2 \end{pmatrix}$

adj A = $(A_c)^T = \begin{pmatrix} 2 & -4 \\ -6 & -3 \end{pmatrix}$

Exercise 1.1 (2) (i) : Find the inverse of $\begin{bmatrix} -2 & 4 \\ 1 & -3 \end{bmatrix}$

Solution: $A = \begin{bmatrix} -2 & 4 \\ 1 & -3 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|}(\text{Adj } A)$

$|A| = \begin{vmatrix} -2 & 4 \\ 1 & -3 \end{vmatrix} = 6 - 4 = 2$

$\text{Adj } A = \begin{bmatrix} -3 & -4 \\ -1 & -2 \end{bmatrix}$

$A^{-1} = \frac{1}{2}\begin{bmatrix} -3 & -4 \\ -1 & -2 \end{bmatrix}$ $(\textit{Using formula})$

Exercise 1.1(9): If adj A = $\begin{pmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{pmatrix}$, find A^{-1}

Solution:

adj A = $\begin{pmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{pmatrix}$

$|\text{adj } A| = \begin{vmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{vmatrix} = 0 + 2(36\text{-}18) + 0 = 2(18) = 36$

$A^{-1} = \pm\frac{1}{\sqrt{|\text{adj } A|}}(\text{adj } A) = \pm\frac{1}{\sqrt{36}}\begin{pmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{pmatrix}$

$= \pm\frac{1}{6}\begin{pmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{pmatrix}$

Exercise 1.2 (1)(i):

Find the rank of the matrix by minor method: $\begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$

Solution:

$A = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$; A is a mattrix of order 2X2; $\rho(A) \leq \min\{2,2\} = 2$

$|A| = \begin{vmatrix} 2 & -4 \\ -1 & 2 \end{vmatrix} = 4 - 4 = 0$; $\rho(A) \neq 2 \Rightarrow \rho(A) < 2$

$a_{11} = 2 \neq 0$ $\Rightarrow$ Since 1 x 1 minor not equal to zero $\rho(A) = 1$

Exercise 1.2 (1)(ii):

Find the rank of the matrix by minor method: $\begin{pmatrix} -1 & 3 \\ 4 & -7 \\ 3 & -4 \end{pmatrix}$

Solution:

$A = \begin{pmatrix} -1 & 3 \\ 4 & -7 \\ 3 & -4 \end{pmatrix}$

A is a mattrix of order 3 X 2; $\rho(A) \leq \min\{3,2\} = 2$

$|A| = \begin{vmatrix} -1 & 3 \\ 4 & -7 \end{vmatrix} = 7 - 12 = -5 \neq 0$;

Since 2 x 2 minor not equal to zero $\Rightarrow \rho(A) = 2$

Exercise 1.2 (1)(iii):

Find the rank of matrix by minor method: $\begin{pmatrix} 1 & -2 & -1 & 0 \\ 3 & -6 & -3 & 1 \end{pmatrix}$

Solution:

$A = \begin{pmatrix} 1 & -2 & -1 & 0 \\ 3 & -6 & -3 & 1 \end{pmatrix}$

A is a mattrix of order 2 X 4; $\rho(A) \leq \min\{4,2\} = 2$

Consider $\Delta_1 = \begin{vmatrix} 1 & -2 \\ 3 & -6 \end{vmatrix} = -6 + 6 = 0$

Consider $\Delta_2 = \begin{vmatrix} -2 & -1 \\ -6 & -3 \end{vmatrix} = 6 - 6 = 0$

We must find all possible 2 x 2 minors of A check $|A| \neq 0$

Consider $\Delta_3 = \begin{vmatrix} -1 & 0 \\ -3 & 1 \end{vmatrix} = -1 - 0 = -1 \neq 0$

Since 2 x 2 minor not equal to zero $\Rightarrow \rho(A) = 2$

Exercise 1.2 (1)(iv):

Find the rank of the matrix by minor method:

$\begin{pmatrix} 1 & -2 & 3 \\ 2 & 4 & -6 \\ 5 & 1 & -1 \end{pmatrix}$

Solution:

$A = \begin{pmatrix} 1 & -2 & 3 \\ 2 & 4 & -6 \\ 5 & 1 & -1 \end{pmatrix}$

A is a mattrix of order 3 X 3; $\rho(A) \leq \min\{3,3\} = 3$

$\Delta_1 = \begin{vmatrix} 1 & -2 & 3 \\ 2 & 4 & -6 \\ 5 & 1 & -1 \end{vmatrix}$

$= 1(-4+6) + 2(-2+30) + 3(2 - 20)$

$= 1(2) + 2(28) + 3(-18) = 2 + 56 - 54 = 4 \neq 0$

Since 3 x 3 minor not equal to zero $\Rightarrow \rho(A) = 3$

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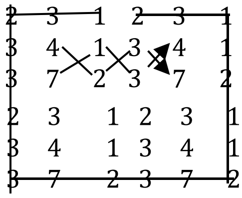
3 MARK QUESTIONS

EXERCISE 1.1

1(ii) Find the adjoint of $\begin{pmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{pmatrix}$ –



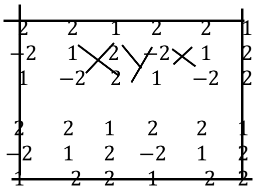
Solution: $A = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{pmatrix}$

$$\text{adj } A = (A_c)^T = \begin{bmatrix} 8-7 & 3-6 & 21-12 \\ 7-6 & 4-3 & 9-14 \\ 3-4 & 3-2 & 8-9 \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & -3 & 9 \\ 1 & 1 & -5 \\ -1 & 1 & -1 \end{bmatrix}^T = \begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix}$$

1(iii).

Find the adjoint of $\frac{1}{3}\begin{pmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{pmatrix}$



Solution: $A = \frac{1}{3}\begin{pmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{pmatrix}$

$$\text{Adj } A = \left(\tfrac{1}{3}\right)^2 \begin{bmatrix} 2+4 & 2+4 & 4-1 \\ -2-4 & 4-1 & 2+4 \\ 4-1 & -4-2 & 2+4 \end{bmatrix}^T$$

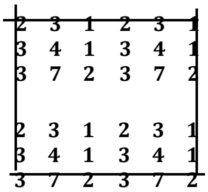
$$= \frac{1}{9}\begin{bmatrix} 6 & 6 & 3 \\ -6 & 3 & 6 \\ 3 & -6 & 6 \end{bmatrix}^T = \frac{1}{9}\begin{bmatrix} 6 & -6 & 3 \\ 6 & 3 & -6 \\ 3 & 6 & 6 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{bmatrix}$$

Exercise 1.1 (2) (iii): Find the inverse of $\begin{pmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{pmatrix}$

Solution: Let $A = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{pmatrix}$

$|A| = 2(8 - 7) - 3(6 - 3) + 1(21 - 12) = 2(1) - 3(3) + 1(9)$
$= 2 - 9 + 9 = 2$ (inverse exists)

$$\text{adj } A = (A_c)^T = \begin{bmatrix} 8-7 & 3-6 & 21-12 \\ 7-6 & 4-3 & 9-14 \\ 3-4 & 3-2 & 8-9 \end{bmatrix}^T$$



$$= \begin{bmatrix} 1 & -3 & 9 \\ 1 & 1 & -5 \\ -1 & 1 & -1 \end{bmatrix}^T = \begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}(\text{Adj } A) = \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix}$$

Practise Ex 1.1 2(ii)

Exercise 1.1 (5): If $A = \frac{1}{9}\begin{pmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{pmatrix}$, prove that $A^{-1} = A^T$

Solution:

$A = \frac{1}{9}\begin{pmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{pmatrix}$ and $A^T = \frac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix}$

$$A A^T = \frac{1}{9}\begin{pmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{pmatrix} \frac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix}$$

$$= \frac{1}{81}\begin{bmatrix} 64+1+16 & -32+4+28 & -8-8+16 \\ -32+4+28 & 16+16+49 & 4-32+28 \\ -8-8+16 & 4-32+28 & 1+64+16 \end{bmatrix}$$

$$= \frac{1}{81}\begin{bmatrix} 81 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 81 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} = I_3$$

$$A^T A = \frac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix} \frac{1}{9}\begin{pmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{pmatrix}$$

$$= \frac{1}{81}\begin{bmatrix} 64+16+1 & -8+16-8 & -32+28+4 \\ -8+16-8 & 1+16+64 & 4+28-32 \\ -32+28+4 & 4+28-32 & 16+49+16 \end{bmatrix}$$

$$= \frac{1}{81}\begin{bmatrix} 81 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 81 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} = I_3$$

$A A^T = A^T A = I_3 \implies A^{-1} = A^T$

Exercise 1.1(6):

If $A = \begin{pmatrix} 8 & -4 \\ -5 & 3 \end{pmatrix}$, verify that $A(\text{adj } A) = (\text{adj } A)A = |A|I_2$

Solution:

$A = \begin{pmatrix} 8 & -4 \\ -5 & 3 \end{pmatrix}$ *and* adj $A = \begin{pmatrix} 3 & 4 \\ 5 & 8 \end{pmatrix}$

$$A(\text{adj } A) = \begin{pmatrix} 8 & -4 \\ -5 & 3 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ 5 & 8 \end{pmatrix} = \begin{pmatrix} 24-20 & 32-32 \\ -15+15 & -20+24 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$$

$$(\text{adj } A)A = \begin{pmatrix} 3 & 4 \\ 5 & 8 \end{pmatrix}\begin{pmatrix} 8 & -4 \\ -5 & 3 \end{pmatrix} = \begin{pmatrix} 24-20 & -12+12 \\ 40-40 & -20+24 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$$

$|A| = \begin{vmatrix} 8 & -4 \\ -5 & 3 \end{vmatrix} = 24 - 20 = 4$

$|A|I_2 = 4\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$

$A(\text{adj } A) = (\text{adj } A)A = |A|I_2 = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$. Hence proved

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Exercise 1.1(9): If adj $A = \begin{pmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{pmatrix}$, find A^{-1}

Solution:

$$\text{adj } A = \begin{pmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{pmatrix}$$

$$|\text{adj } A| = \begin{vmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{vmatrix} = 0 + 2(36-18) + 0 = 2(18) = 36$$

$$A^{-1} = \pm \frac{1}{\sqrt{|\text{ adj } A|}}(\text{adj } A)$$

$$= \pm \frac{1}{\sqrt{36}}\begin{pmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{pmatrix}$$

$$= \pm \frac{1}{6}\begin{pmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{pmatrix}$$

Exercise 1.1 (7): $A = \begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & -3 \\ 5 & 2 \end{pmatrix}$.

verify that $(AB)^{-1} = B^{-1}A^{-1}$

Solution: $A = \begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix}$, $B = \begin{pmatrix} -1 & -3 \\ 5 & 2 \end{pmatrix}$

$$AB = \begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix}\begin{pmatrix} -1 & -3 \\ 5 & 2 \end{pmatrix} = \begin{pmatrix} -3+10 & -9+4 \\ -7+25 & -21+10 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & -5 \\ 18 & -11 \end{pmatrix}$$

$$|AB| = \begin{vmatrix} 7 & -5 \\ 18 & -11 \end{vmatrix} = -77+90 = 13$$

$$\text{adj}(AB) = \begin{pmatrix} -11 & 5 \\ -18 & 7 \end{pmatrix}$$

$$(AB)^{-1} = \frac{1}{|AB|}(\text{Adj } AB) = \frac{1}{13}\begin{pmatrix} -11 & 5 \\ -18 & 7 \end{pmatrix}$$

$$B = \begin{pmatrix} -1 & -3 \\ 5 & 2 \end{pmatrix} \text{ and } |B| = \begin{vmatrix} -1 & -3 \\ 5 & 2 \end{vmatrix} = -2+15 = 13$$

$$\text{adj } B = \begin{pmatrix} 2 & 3 \\ -5 & -1 \end{pmatrix}$$

$$B^{-1} = \frac{1}{|B|}(\text{Adj } B) = \frac{1}{13}\begin{pmatrix} 2 & 3 \\ -5 & -1 \end{pmatrix}$$

$$A = \begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix} \text{ and } |A| = \begin{vmatrix} 3 & 2 \\ 7 & 5 \end{vmatrix} = 15-14 = 1$$

$$\text{adj } A = \begin{pmatrix} 5 & -2 \\ -7 & 3 \end{pmatrix}$$

$$A^{-1} = \frac{1}{|A|}(\text{Adj } A) = \frac{1}{1}\begin{pmatrix} 5 & -2 \\ -7 & 3 \end{pmatrix}$$

$$B^{-1}A^{-1} = \frac{1}{13}\begin{pmatrix} 2 & 3 \\ -5 & -1 \end{pmatrix}\frac{1}{1}\begin{pmatrix} 5 & -2 \\ -7 & 3 \end{pmatrix} = \frac{1}{13}\begin{pmatrix} 2 & 3 \\ -5 & -1 \end{pmatrix}\begin{pmatrix} 5 & -2 \\ -7 & 3 \end{pmatrix}$$

$$= \frac{1}{13}\begin{pmatrix} 10-21 & -4+9 \\ -25+7 & 10-3 \end{pmatrix} = \frac{1}{13}\begin{pmatrix} -11 & 5 \\ -18 & 7 \end{pmatrix}$$

$(AB)^{-1} = B^{-1}A^{-1} = \frac{1}{13}\begin{pmatrix} -11 & 5 \\ -18 & 7 \end{pmatrix}$ Hence proved.

Exercise 1.1(8): If $\text{adj}A = \begin{pmatrix} 2 & -4 & 2 \\ -3 & 12 & -7 \\ -2 & 0 & 2 \end{pmatrix}$, find A

Solution: $A = \pm\frac{1}{\sqrt{|\text{adj } A|}}\text{adj(adj } A)$

$$|\text{Adj } A| = \begin{vmatrix} 2 & -4 & 2 \\ -3 & 12 & -7 \\ -2 & 0 & 2 \end{vmatrix}$$

$$\begin{array}{cccccc} 2 & -4 & 2 & 2 & -4 & 2 \\ -3 & 12 & -7 & -3 & 12 & -7 \\ -2 & 0 & 2 & -2 & 0 & 2 \\ 2 & -4 & 2 & 2 & -4 & 2 \\ -3 & 12 & -7 & -3 & 12 & -7 \\ -2 & 0 & 2 & -2 & 0 & 2 \end{array}$$

$= 2(24-0)-(-4)(-6-14)+2(0+24)$

$= 2(24) + 4(-20) + 2(24)$

$= 48 - 80 + 48 = 96 - 80 = 16$

$$\text{adj(adj } A) = \begin{bmatrix} 24-0 & 14+6 & 0+24 \\ 0+8 & 4+4 & 8-0 \\ 28-24 & -6+14 & 24-12 \end{bmatrix}^T$$

$$= \begin{bmatrix} 24 & 20 & 24 \\ 8 & 8 & 8 \\ 4 & 8 & 12 \end{bmatrix}^T = \begin{bmatrix} 24 & 8 & 4 \\ 20 & 8 & 8 \\ 24 & 8 & 12 \end{bmatrix}$$

$$A = \pm\frac{1}{\sqrt{16}}\begin{bmatrix} 24 & 8 & 4 \\ 20 & 8 & 8 \\ 24 & 8 & 12 \end{bmatrix}$$

$$= \pm\frac{1}{4}4\begin{bmatrix} 6 & 2 & 1 \\ 5 & 2 & 2 \\ 6 & 2 & 3 \end{bmatrix}$$

$$= \pm\begin{bmatrix} 6 & 2 & 1 \\ 5 & 2 & 2 \\ 6 & 2 & 3 \end{bmatrix}$$

Exercise 1.1(10): Find adj(adj A), if $\text{adj } A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}$

SOLUTION:

$$\text{adj } A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$\begin{array}{cccccc} 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 2 & 0 & 0 & 2 & 0 \\ -1 & 0 & 1 & -1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 2 & 0 & 0 & 2 & 0 \\ -1 & 0 & 1 & -1 & 0 & 1 \end{array}$$

$$\text{Adj(adj } A) = \begin{bmatrix} 2-0 & 0-0 & 0+2 \\ 0-0 & 1+1 & 0-0 \\ 0-2 & 0-0 & 2-0 \end{bmatrix}^T$$

$$= \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ -2 & 0 & 2 \end{bmatrix}^T = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

Exercise 1.1(11): $A = \begin{pmatrix} 1 & \tan x \\ -\tan x & 1 \end{pmatrix}$, show that

$A^TA^{-1} = \begin{pmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{pmatrix}$

SOLUTION: $|A| = 1 + \tan^2 x = \sec^2 x$

$$\text{Adj } A = \begin{pmatrix} 1 & -\tan x \\ \tan x & 1 \end{pmatrix} \quad \text{and} \quad A^T = \begin{pmatrix} 1 & -\tan x \\ \tan x & 1 \end{pmatrix}$$

$$A^{-1} = \frac{1}{|A|}(\text{Adj } A)$$

$$A^{-1} = \frac{1}{\sec^2 x|}\begin{pmatrix} 1 & -\tan x \\ \tan x & 1 \end{pmatrix}$$

$$A^TA^{-1} = \frac{1}{\sec^2 x}\begin{pmatrix} 1 & -\tan x \\ \tan x & 1 \end{pmatrix}\begin{pmatrix} 1 & -\tan x \\ \tan x & 1 \end{pmatrix}$$

$$= \cos^2 x\begin{pmatrix} 1-\tan^2 x & -\tan x - \tan x \\ \tan x + \tan x & -\tan^2 x + 1 \end{pmatrix}$$

$$= \cos^2 x\begin{pmatrix} 1-\frac{\sin^2 x}{\cos^2 x} & -2\tan x \\ 2\tan x & 1-\frac{\sin^2 x}{\cos^2 x} \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2 x - \frac{\sin^2 x\cos^2 x}{\cos^2 x} & -2\tan x\cos^2 x \\ 2\tan x\cos^2 x & \cos^2 x - \frac{\sin^2 x\cos^2 x}{\cos^2 x} \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2 x - \sin^2 x & -2\sin x\cos x \\ 2\sin x\cos x & \cos^2 x - \sin^2 x \end{pmatrix}$$

$$= \begin{pmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{pmatrix}$$

Exercise 1.1(12):

Find the matrix A for which $A\begin{pmatrix} 5 & 3 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} 14 & 7 \\ 7 & 7 \end{pmatrix}$

SOLUTION:

$$A\begin{pmatrix} 5 & 3 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} 14 & 7 \\ 7 & 7 \end{pmatrix}$$

$$A \quad B \quad = C \Rightarrow A = C\,B^{-1}$$

$$B = \begin{pmatrix} 5 & 3 \\ -1 & -2 \end{pmatrix} \Rightarrow |B| = -10+3 = -7$$

$$\text{Adj } B = \begin{pmatrix} -2 & -3 \\ 1 & 5 \end{pmatrix} \Rightarrow B^{-1} = \frac{1}{-7}\begin{pmatrix} -2 & -3 \\ 1 & 5 \end{pmatrix}$$

$$A = C\,B^{-1} = \frac{1}{-7}\begin{pmatrix} 14 & 7 \\ 7 & 7 \end{pmatrix}\begin{pmatrix} -2 & -3 \\ 1 & 5 \end{pmatrix}$$

$$= \frac{1}{-7}\begin{pmatrix} -28+7 & -42+35 \\ -14+7 & -21+35 \end{pmatrix}$$

$$= \frac{1}{-7}\begin{pmatrix} -21 & -7 \\ -7 & 14 \end{pmatrix} = \begin{pmatrix} \frac{-21}{-7} & \frac{-7}{-7} \\ \frac{-7}{-7} & \frac{14}{-7} \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & -2 \end{pmatrix}$$

Exercise 1.1(13):

Given $A = \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix}$, and $C = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$,

Find a matrix X such that AXB = C.

Solution:

$AXB = C \Rightarrow A^{-1}(AXB)B^{-1} = A^{-1}C\,B^{-1}$

$\Rightarrow (A^{-1}A)X(BB^{-1}) = A^{-1}C\,B^{-1}$

$\Rightarrow X = A^{-1}C\,B^{-1}$

$A = \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix} \Rightarrow |A| = 0+2 = 2$ & $\text{Adj } A = \begin{pmatrix} 0 & 1 \\ -2 & 1 \end{pmatrix}$

$A^{-1} = \frac{1}{|A|}(\text{Adj } A) = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -2 & 1 \end{pmatrix}$

$B = \begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix} \Rightarrow |B| = 3+2 = 5$ & $\text{Adj } B = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$

$B^{-1} = \frac{1}{|B|}(\text{Adj } B) = \frac{1}{5}\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$

$X = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}\frac{1}{5}\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$

$= \frac{1}{10}\begin{pmatrix} 0 & 1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$

$= \frac{1}{10}\begin{pmatrix} 0+2 & 0+2 \\ -2+2 & -2+2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$

$= \frac{1}{10}\begin{pmatrix} 2-2 & 4+6 \\ 0+0 & 0+0 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 0 & 10 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

Exercise 1.1 (14): If $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$, show that $A^{-1} = \frac{1}{2}(A^2 - 3I)$

Solution:

$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

0	1	1	0	1	1
1	0	1	1	0	1
1	1	0	1	1	0
0	1	1	0	1	1
1	0	1	1	0	1
1	1	0	1	1	0

$\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 0\,(0-1) - 1\,(0-1) + 1(1-0)$

$= 0(-1) - 1(-1) + 1(1)$

$= 0 + 1 + 1 = 2$

$\text{Adj } A = (A_c)^T = \begin{bmatrix} 0-1 & 1-0 & 1-0 \\ 1-0 & 0-1 & 1-0 \\ 0-1 & 1-0 & 0-1 \end{bmatrix}^T = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix}^T$

$= \begin{bmatrix} -1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|}(\text{Adj } A) = \frac{1}{2}\begin{bmatrix} -1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

$A^2 = A \times A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

$= \begin{bmatrix} 0+1+1 & 0+0+1 & 0+1+0 \\ 0+0+1 & 1+0+1 & 1+0+0 \\ 0+1+0 & 1+0+0 & 1+1+0 \end{bmatrix}$

$= \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

$A^2 - 3I = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} + \begin{pmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix}$

$= \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$

$\frac{1}{2}(A^2 - 3I) = \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} = A^{-1}$

Exercise 1.2 (2) (i):

Find the rank of the matrix by row reduction method:

$\begin{pmatrix} 1 & 1 & 1 & 3 \\ 2 & -1 & 3 & 4 \\ 5 & -1 & 7 & 11 \end{pmatrix}$

Solution:

$A = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & -1 & 3 & 4 \\ 5 & -1 & 7 & 11 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -3 & 1 & -2 \\ 0 & -6 & 2 & -4 \end{bmatrix} R_2 \to R_2 - 2R_1;\ R_3 \to R_3 - 5R_1$

$\sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -3 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 - 2R_2$

This is in echelon form; no of nonzero rows = 2 $\Longrightarrow \rho(A) = 2$

Exercise 1.2 (2) (ii):

Find the rank of the matrices by row reduction method:

$\begin{pmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 1 & -2 & 3 \\ 1 & -1 & 1 \end{pmatrix}$

Solution:

$A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 1 & -2 & 3 \\ 1 & -1 & 1 \end{pmatrix}$

$\sim \begin{pmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & -4 & 4 \\ 0 & -3 & 2 \end{pmatrix} R_2 \to R_2 - 3R_1;\ R_3 \to R_3 - R_1;\ R_4 \to R_4 - R_1$

$\sim \begin{pmatrix} 1 & 2 & -1 \\ 0 & -84 & 60 \\ 0 & -84 & 84 \\ 0 & -84 & 56 \end{pmatrix} R_2 \to 12R_2;\ R_3 \to 21R_3;\ R_4 \to 28R_4$

$\sim \begin{pmatrix} 1 & 2 & -1 \\ 0 & -84 & 60 \\ 0 & 0 & 24 \\ 0 & 0 & -4 \end{pmatrix} R_3 \to R_3 - R_2;\ R_4 \to R_4 - R_2$

$\sim \begin{pmatrix} 1 & 2 & -1 \\ 0 & -84 & 60 \\ 0 & 0 & 4 \\ 0 & 0 & -4 \end{pmatrix} R_3 \to \frac{1}{6}R_3$

$\sim \begin{pmatrix} 1 & 2 & -1 \\ 0 & -84 & 60 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix} R_4 \to R_4 + R_3$

This is in echelon form; no of nonzero rows = 3 $\Longrightarrow \rho(A) = 3$

Exercise 1.2 (2) (iii):

Find the rank of the matrices by row reduction method:

$$\begin{pmatrix} 3 & -8 & 5 & 2 \\ 2 & -5 & 1 & 4 \\ -1 & 2 & 3 & -2 \end{pmatrix}$$

Solution:

$$A = \begin{pmatrix} 3 & -8 & 5 & 2 \\ 2 & -5 & 1 & 4 \\ -1 & 2 & 3 & -2 \end{pmatrix} \sim \begin{bmatrix} -1 & 2 & 3 & -2 \\ 2 & -5 & 1 & 4 \\ 3 & -8 & 5 & 2 \end{bmatrix} R_1 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} -1 & 2 & 3 & -2 \\ 0 & -1 & 7 & 0 \\ 0 & -2 & 14 & -4 \end{bmatrix} R_2 \to R_2 + 2R_1;\ R_3 \to R_3 + 3R_1$$

$$\sim \begin{bmatrix} -1 & 2 & 3 & -2 \\ 0 & -1 & 7 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix} R_3 \to R_3 - 2R_2$$

$\Rightarrow \rho(A) = 3$

Exercise 1.2 (3)(i):

Find the inverse of the matrix by Gauss Jordan method: $\begin{pmatrix} 2 & -1 \\ 5 & -2 \end{pmatrix}$

Solution: Let $A = \begin{pmatrix} 2 & -1 \\ 5 & -2 \end{pmatrix}$

$$[A|I] = \left(\begin{array}{cc|cc} 2 & -1 & 1 & 0 \\ 5 & -2 & 0 & 1 \end{array}\right) \qquad \text{l.. C. M. of 2 and 5 is 10}$$

$$\sim \left(\begin{array}{cc|cc} 10 & -5 & 5 & 0 \\ 10 & -4 & 0 & 2 \end{array}\right) R_1 \to 5R_1;\ R_2 \to 2R_2$$

$$\sim \left(\begin{array}{cc|cc} 10 & -5 & 5 & 0 \\ 0 & 1 & -5 & 2 \end{array}\right) R_2 \to R_2 - R_1$$

$$\sim \left(\begin{array}{cc|cc} 2 & -1 & 1 & 0 \\ 0 & 1 & -5 & 2 \end{array}\right) R_1 \to 5R_1$$

$$\sim \left(\begin{array}{cc|cc} 2 & 0 & -4 & 2 \\ 0 & 1 & -5 & 2 \end{array}\right) R_1 \to R_1 + R_2$$

$$\sim \left(\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & -5 & 2 \end{array}\right) R_1 \to \tfrac{1}{2}R_1$$

$\therefore\ A^{-1} = \begin{pmatrix} -2 & 1 \\ -5 & 2 \end{pmatrix}$

Exercise 1.3(1)(i):

Solve the following system of linear equations by matrix inversion method: $2x + 5y = -2, x + 2y = -3$

SOLUTION: $2x + 5y = -2, x + 2y = -3$

$$\begin{pmatrix} 2 & 5 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$$

A X = B $\Rightarrow X = A^{-1}B$

$A = \begin{pmatrix} 2 & 5 \\ 1 & 2 \end{pmatrix}, X = \begin{pmatrix} x \\ y \end{pmatrix} B = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$

$|A| = \begin{vmatrix} 2 & 5 \\ 1 & 2 \end{vmatrix} = 4 - 5 = -1 \neq 0$, A^{-1} exists

adj $A = \begin{pmatrix} 2 & -5 \\ -1 & 2 \end{pmatrix}$

$A^{-1} = \frac{1}{|A|}(\text{adj } A) = \frac{1}{-1}\begin{pmatrix} 2 & -5 \\ -1 & 2 \end{pmatrix}$

$X = A^{-1}B$

$$\begin{pmatrix} x \\ y \end{pmatrix} = -1\begin{pmatrix} 2 & -5 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} -2 \\ -3 \end{pmatrix} = -1\begin{pmatrix} -4+15 \\ 2-6 \end{pmatrix} = -1\begin{pmatrix} 11 \\ -4 \end{pmatrix}$$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -11 \\ 4 \end{pmatrix} \Rightarrow x = -11,\ y = 4$

Exercise 1.3(1)(ii):

Solve the following system of linear equations by matrix inversion method : $2x - y = 8,\ 3x + 2y = -2$

SOLUTION: $2x - y = 8,\ 3x + 2y = -2$

$$\begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ -2 \end{pmatrix} \Rightarrow AX = B \Rightarrow X = A^{-1}B$$

$A = \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}, X = \begin{pmatrix} x \\ y \end{pmatrix}, B = \begin{pmatrix} 8 \\ -2 \end{pmatrix}$

$|A| = \begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} = 4 + 3 = 7 \neq 0$, A^{-1} exists

adj $A = \begin{pmatrix} 2 & 1 \\ -3 & 2 \end{pmatrix} \Rightarrow A^{-1} = \frac{1}{|A|}(\text{adj } A) = \frac{1}{7}\begin{pmatrix} 2 & 1 \\ -3 & 2 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 2 & 1 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 8 \\ -2 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 16-2 \\ -24-4 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 14 \\ -28 \end{pmatrix} = \begin{pmatrix} \frac{14}{7} \\ \frac{-28}{7} \end{pmatrix}$$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix} \Rightarrow x = 2,\ y = -4$

Exercise 1.3(4):

Four men and 4 women can finish a piece of work jointly in 3 days while 2 men and 5 women can finish the same work jointly in 4 days. Find the time taken by one man alone and that of one woman alone to finish the same work by using matrix inversion method.

SOLUTION:

Let one man complete the work in x days

let one women complete the work in y days

man one day work $= \frac{1}{x}$, women one day work $= \frac{1}{y}$

Given: $4(\frac{1}{x}) + 4(\frac{1}{y}) = \frac{1}{3},\ 2(\frac{1}{x}) + 5(\frac{1}{y}) = \frac{1}{4}$

$$\begin{pmatrix} 4 & 4 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} \frac{1}{x} \\ \frac{1}{y} \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{4} \end{pmatrix} \Rightarrow AX = B \Rightarrow X = A^{-1}B$$

$A = \begin{pmatrix} 4 & 4 \\ 2 & 5 \end{pmatrix}, X = \begin{pmatrix} \frac{1}{x} \\ \frac{1}{y} \end{pmatrix};\ B = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{4} \end{pmatrix}$

$|A| = \begin{vmatrix} 4 & 4 \\ 2 & 5 \end{vmatrix} = 20 - 8 = 12 \neq 0$, A^{-1} exists

adj $A = \begin{pmatrix} 5 & -4 \\ -2 & 4 \end{pmatrix} \Rightarrow A^{-1} = \frac{1}{12}\begin{pmatrix} 5 & -4 \\ -2 & 4 \end{pmatrix}$

$$X = \frac{1}{12}\begin{pmatrix} 5 & -4 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} \frac{1}{3} \\ \frac{1}{4} \end{pmatrix} = \frac{1}{12}\begin{pmatrix} \frac{5}{3} + \frac{-4}{4} \\ \frac{-2}{3} + \frac{4}{4} \end{pmatrix} = \frac{1}{12}\begin{pmatrix} \frac{20-12}{12} \\ \frac{-8+12}{12} \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{x} \\ \frac{1}{y} \end{pmatrix} = \frac{1}{12}\begin{pmatrix} \frac{8}{12} \\ \frac{4}{12} \end{pmatrix} = \begin{pmatrix} \frac{8}{144} \\ \frac{4}{144} \end{pmatrix} = \begin{pmatrix} \frac{1}{18} \\ \frac{1}{36} \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{x} \\ \frac{1}{y} \end{pmatrix} = \begin{pmatrix} \frac{1}{18} \\ \frac{1}{36} \end{pmatrix} \Rightarrow \frac{1}{x} = \frac{1}{18} \Rightarrow x = 18 \text{ and } \frac{1}{y} = \frac{1}{36} \Rightarrow y = 36$$

One man can complete the work in 18 days

one woman can complete the work in 36 days

Exercise 1.3(3):

A man is appointed in a job with a monthly salary of certain amount and a fixed amount of annual increment. If his salary was ₹19,800 per month at the end of the first month after 3 years of service and ₹23,400 per month at the end of the first month after 9 years of service, find his starting salary and his annual increment.

(Use matrix inversion method to solve the problem.)

SOLUTION:

Let salary be ₹ x and annual increment be ₹ y

Given: $x + 3y = 19800$ and $x + 9y = 23400$

$$\begin{pmatrix}1 & 3\\1 & 9\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}19800\\23400\end{pmatrix}$$

$AX = B \Rightarrow X = A^{-1}B$ and $A^{-1} = \frac{1}{|A|}(\text{adj } A)$

$|A| = \begin{vmatrix}1 & 3\\1 & 9\end{vmatrix} = 9 - 3 = 6 \neq 0$, A^{-1} exists

adj A = $\begin{pmatrix}9 & -3\\-1 & 1\end{pmatrix}$ & $A^{-1} = \frac{1}{6}\begin{pmatrix}9 & -3\\-1 & 1\end{pmatrix}$

$$X = \frac{1}{6}\begin{pmatrix}9 & -3\\-1 & 1\end{pmatrix}\begin{pmatrix}19800\\23400\end{pmatrix}$$

$$= \frac{1}{6}\begin{pmatrix}178200 - 70200\\-19800 + 23400\end{pmatrix}$$

$$= \frac{1}{6}\begin{pmatrix}108000\\3600\end{pmatrix}$$

$$= \begin{pmatrix}108000/6\\3600/6\end{pmatrix} = \begin{pmatrix}18000\\600\end{pmatrix}$$

Initial salary x = ₹ 18000, annual increment = ₹ 600

Exercise 1.4(1)(i):

Solve : $5x - 2y + 16 = 0, x + 3y - 7 = 0$

Solution: $5x - 2y + 16 = 0, x + 3y - 7 = 0$

$5x - 2y = -16, x + 3y = 7$

$\Delta = \begin{vmatrix}5 & -2\\1 & 3\end{vmatrix} = 15 + 2 = 17$

$\Delta_x = \begin{vmatrix}-16 & -2\\7 & 3\end{vmatrix} = -48 + 14 = -34$

$\Delta_y = \begin{vmatrix}5 & -16\\1 & 7\end{vmatrix} = 35 + 16 = 51$

$x = \frac{\Delta_x}{\Delta} = \frac{-34}{17} = -2$, $x = -2$

$y = \frac{\Delta_y}{\Delta} = \frac{51}{17} = 3$, $y = 3$

Exercise 1.4(1)(ii): Solve the following systems of linear equations by Cramer's rule: $\frac{3}{x} + 2y = 12$, $\frac{2}{x} + 3y = 13$

Solution:

$\Delta = \begin{vmatrix}3 & 2\\2 & 3\end{vmatrix} = 9 - 4 = 5$

$\Delta_{\frac{1}{x}} = \begin{vmatrix}12 & 2\\13 & 3\end{vmatrix} = 36 - 26 = 10$

$\Delta_y = \begin{vmatrix}3 & 12\\2 & 13\end{vmatrix} = 39 - 24 = 15$

$\frac{1}{x} = \frac{10}{5} = 2$, $y = \frac{15}{5} = 3$

$x = \frac{1}{2}$, $y = 3$

Exercise 1.4(2):

In a competitive examination, one mark is awarded for every correct answer while $\frac{1}{4}$ mark is deducted for every wrong answer. A student answered 100 questions and got 80 marks. How many questions did he answer correctly ? (Use Cramer's rule to solve the problem).

Solution: Let Number of question answered correctly be x

Let Number of question answered wrong be y

Given : For correct answer 1 mark, wrong answer $-\frac{1}{4}$ mark

$x + y = 100$; $x - \frac{1}{4}y = 80 \Rightarrow 4x - y = 320$

$\Delta = \begin{vmatrix}1 & 1\\4 & -1\end{vmatrix} = -1 - 4 = -5$

$\Delta_x = \begin{vmatrix}100 & 1\\320 & -1\end{vmatrix} = -100 - 320 = -420$

$\Delta_y = \begin{vmatrix}1 & 100\\4 & 320\end{vmatrix} = 320 - 400 = -80$

$X = \frac{\Delta_x}{\Delta} = \frac{-420}{-5} = 84$, No. of questions answered correctly = 84

$Y = \frac{\Delta_y}{\Delta} = \frac{-80}{-5} = 16$. No. of question answered wrong = 16

Exercise 1.4(3):

A chemist has one solution which is 50% acid and another solution which is 25% acid. How much each should be mixed to make 10 litres of a 40% acid solution? (Use Cramer's Rule)

Solution: Let 50% acid be x litres and 25% acid be y litres

Given: $x + y = 10$

$\frac{50}{100}x + \frac{25}{100}y = \frac{40}{100}(10) \Rightarrow 10x + 5y = 80$

$x + y = 10$; $10x + 5y = 80$

$\Delta = \begin{vmatrix}1 & 1\\10 & 5\end{vmatrix} = 5 - 10 = -5$

$\Delta_x = \begin{vmatrix}10 & 1\\80 & 5\end{vmatrix} = 50 - 80 = -30$

$\Delta_y = \begin{vmatrix}1 & 10\\10 & 80\end{vmatrix} = 80 - 100 = -20$

$x = \frac{\Delta_x}{\Delta} = \frac{-30}{-5} = 6$, 50% acid 6 litres to be mixed

$y = \frac{\Delta_y}{\Delta} = \frac{-20}{-5} = 4$, 25% acid 4 litres to be mixed

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Exerscise 1.4(4):

A fish tank can be filled in 10 minutes using both pumps A and B simultaneously. However, Pump B can pump water in or out at the same rate. If Pump B is dvertently run in reverse, then the tank will be filled in 30 minutes. How long would it take each pump to fill the tank by it self

SOLUTION:

Let the Pump A and Pump B fill the tank in x and y mins.

Water filled by Pump A and Pump B in 1 min is $\frac{1}{x}, \frac{1}{y}$ resp.

Given: $\frac{1}{x} + \frac{1}{y} = \frac{1}{10}$ and $\frac{1}{x} - \frac{1}{y} = \frac{1}{30}$

$\Rightarrow \frac{10}{x} + \frac{10}{y} = 1$ and $\frac{30}{x} - \frac{30}{y} = 1$

$\Delta = \begin{vmatrix}10 & 10\\30 & -30\end{vmatrix} = -300 - 300 = -600$

$\Delta_{\frac{1}{x}} = \begin{vmatrix}1 & 10\\1 & -30\end{vmatrix} = -30 - 10 = -40$

$\Delta_{\frac{1}{y}} = \begin{vmatrix}10 & 1\\30 & 1\end{vmatrix} = 10 - 30 = -20$

$\frac{1}{x} = \frac{-40}{-600} = \frac{1}{15}$, Pump A fill the tank in 15 mins

$\frac{1}{y} = \frac{-20}{-600} = \frac{1}{30}$, Pump B fill the tank in 30 mins

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Exercise 1.6(1)(iii):

Test for consistency and if possible, solve the following systems of equations by rank method:

$2x + 2y + z = 5, x - y + z = 1, 3x + y + 2z = 4$

Solution:

$$\begin{pmatrix} 2 & 2 & 1 \\ 1 & -1 & 1 \\ 3 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 4 \end{pmatrix}$$

A X = B

$$[A|B] = \begin{bmatrix} 2 & 2 & 1 & 5 \\ 1 & -1 & 1 & 1 \\ 3 & 1 & 2 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 1 \\ 2 & 2 & 1 & 5 \\ 3 & 1 & 2 & 4 \end{bmatrix} R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 4 & -1 & 3 \\ 0 & 4 & -1 & 1 \end{bmatrix} R_2 \to R_2 - 2R_1 ; R_3 \to R_3 - 3R_1$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 4 & -1 & 3 \\ 0 & 0 & 0 & -2 \end{bmatrix} R_3 \to R_3 - R_2$$

$\rho([A|B]) = 3$, $\rho(A) = 2$

$\rho([A|B]) \neq \rho(A)$

System inconsistent , No solution

Exercise 1.7(1)(ii):

Solve the following system of homogenous equations:

$2x + 3y - z = 0, x - y - 2z = 0, 3x + y + 3z = 0$

Solution:

$2x + 3y - z = 0, x - y - 2z = 0, 3x + y + 3z = 0$

$$\begin{pmatrix} 2 & 3 & -1 \\ 1 & -1 & -2 \\ 3 & 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

A X = O

$$[A|O] = \begin{bmatrix} 2 & 3 & -1 & 0 \\ 1 & -1 & -2 & 0 \\ 3 & 1 & 3 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & 0 \\ 2 & 3 & -1 & 0 \\ 3 & 1 & 3 & 0 \end{bmatrix} R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & 0 \\ 0 & 5 & 3 & 0 \\ 0 & 4 & 9 & 0 \end{bmatrix} R_2 \to R_2 - 2R_1; R_3 \to R_3 - 3R_1$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & 0 \\ 0 & 20 & 12 & 0 \\ 0 & 20 & 45 & 0 \end{bmatrix} R_2 \to 4R_2 ; R_3 \to 5R_3$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & 0 \\ 0 & 20 & 12 & 0 \\ 0 & 0 & 33 & 0 \end{bmatrix} R_3 \to R_3 - R_2$$

⇒ $\rho([A|O]) = 3$, $\rho(A) = 3$

⇒ $\rho([A|O]) = \rho(A) = 3$ = Number of unknowns

⇒ system consistent with unique solution

⇒ System consistent with trivial solution

⇒ $x = 0, y = 0, z = 0$

Question 15.

Decrypt the received encoded message $[2 \quad -3][20 \quad 4]$ with the encryption matrix $\begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$ and the decryption matrix as its inverse, where the system of codes are described by the numbers $1 - 26$ to the letters A- Z respectively, and the number 0 to a blank space.

Solution:

Let the encoding matrix be $\begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$

Let $A = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}$

$|A| = \begin{vmatrix} -1 & -1 \\ 2 & 1 \end{vmatrix} = -1 + 2 = 1$

Now adj $A = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}$

So $A^{-1} = \frac{1}{1}\begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}$

Now coded row matrix (B)	Decoded row matrix (BA^{-1})
$(2 \quad -3)$	$\overrightarrow{(2 \quad -3)}\begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} = (2+6 \;\; 2+3) = (8 \quad 5)$
$(20 \quad 4)$	$(20 \quad 4)\begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} = (20-8 \;\; 20-4) = (12 \quad 16)$

So the sequence of decoded matrices is [8 5] , [12 16]. Thus the receivers read this message as HELP.

CHAPTER 2 : COMPLEX NUMBERS

2 MARKS, 3 MARKS, 5 MARKS

2 MARKS

Ex 2.1

Simplify:

(i) $i^{1947} + i^{1950}$ $\quad 1947 = 1944 + 3$

$= i^{1944} \cdot i^3 + i^{1948} \cdot i^2$ $\quad 1950 = 1948 + 2$

$= i^3 + i^2 = - i - 1$ $\quad i^{1944} = 1$

$= - 1 - i$ $\quad i^{1948} = 1$

---

2. $i^{1948} - i^{-1869}$ $\quad 1948 = $ multiple of 4

$= i^{1948} - \frac{1}{i^{1869}}$ $\quad 1869 = 1868 + 1$

$= 1 - \frac{1}{i^{1869}} = 1 - \frac{1}{i^{1868+1}} = 1 - \frac{1}{i^{1868} \cdot i^1}$

$= 1 - \frac{1}{i} = 1 - \frac{1}{i} \times \frac{i}{i} = 1 - \frac{i}{i^2}$

$= 1 - (- i) = 1 + i$

---

3. $\sum_{n=1}^{12} i^{12} = i^1 + i^2 + i^3 + i^4 + i^5 + i^6 + i^7 + i^8 + i^9 + i^{10} + i^{11} + i^{12}$

$= i - 1 - i + 1 + i.i^4 + i^4 i^2 + i^4 i^3 + i^4 i^4 + i^8 i + i^8 i^2 + i^8 i^3 + i^8 i^4$

$= i - 1 - i + 1 + i - 1 - i + 1 + i - 1 - i + 1 = 0$

---

4. $i^{59} + \frac{1}{i^{69}} = i^{56} \cdot i^3 + \frac{1}{i^{56} i^3}$ $\quad \left(\because \frac{1}{i^3} = i\right)$

$= - i + \frac{1}{1 \cdot (i)^3} = - i + i = 0$

---

5. $i \cdot i^2 \cdot i^3 \dots i^{2000} = i^{1+2+3+\dots+2000}$ $\quad \sum n = \frac{n(n+1)}{2}$

$= i^{\frac{2000(2000+1)}{2}} = i^{1000 \times 2001}$

$= 1$ $\quad [\because i^{(\text{multiple of } 4)} = 1.]$

---

6. $\sum_{n=1}^{10} i^{n+50}$ $\quad i^n + i^{n+1} + i^{n+2} + i^{n+3} = 0$

$= (i^{51} + i^{52} + i^{53} + i^{54}) + (i^{55} + i^{56} + i^{57} + i^{58} + i^{59} + i^{60})$

$= 0 + 0 + i^{56} \cdot i^3 + 1$

$= - i + 1 = 1 - i$

Ex 2.2

(1) $z = 5 - 2i$ $\quad w = - 1 + 3i$ Find the value of

(i) $z + w = 5 - 2i + (- 1 + 3i) = 5 - 2i - 1 + 3i = 4 + i$

(ii) $z - iw = 5 - 2i - i(- 1 + 3i) = 5 - 2i + i - 3i^2$

$= 5 - i - 3(- 1) = 5 - i + 3 = 8 - i$

(iii) $2z + 3w = 2(5 - 2i) + 3(- 1 + 3i) = 10 - 4i - 3 + 9i$

$= 7 + 5i$

(iv) $zw = (5 - 2i)(- 1 + 3i) = - 5 + 15i + 2i - 6i^2$

$= - 5 + 17i + 6 = 1 + 17i$

(v) $z^2 + 2zw + w^2 = (z + w)^2 = (4 + i)^2$ (Ref(i))

$= 4^2 + 2(4)i + i^2 = 16 + 8i - 1 = 15 + 8i$

(vi) $(z + w)^2 = (4 + i)^2 = 16 + 8i - 1 = 15 + 8i$

EXERCISE 2.3

1. $z_1 = 1 - 3i$, $z_2 = - 4i$, $z_3 = 5$

(i) $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$

L.H.S $= (z_1 + z_2) + z_3$

$= [1 - 3i + (-4i)] + 5 = (1 - 3i - 4i) + 5$

$= 1 - 7i + 5 = 6 - 7i$ $\quad$ - (1)

R.H.S $= z_1 + (z_2 + z_3)$

$= 1 - 3i + (-4i + 5)$

$= 1 - 3i - 4i + 5$

$= 6 - 7i$ $\quad$ - (2)

(1) = (2) $\quad$ LHS = RHS

$\therefore (z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$

(ii) $(z_1 z_2) z_3 = z_1 (z_2 z_3)$

L.H.S : $z_1 z_2 = (1 - 3i)(- 4i) = - 4i + 12i^2 = - 4i - 12$

$(z_1 z_2) z_3 = (-12 - 4i)5 = - 60 - 20i$ $\quad$ - (3)

R.H.S : $z_2 z_3 = (- 4i)5 = - 20i$

$z_1(z_2 z_3) = (1 - 3i)(- 20i) = - 20i + 60i^2$

$= - 60 - 20i$ $\quad$ - (4)

(3) =(4) $\quad (z_1 z_2) z_3 = z_1 (z_2 z_3)$

EXERCISE 2.4

1. Write in the rectangular form.

(1) $\overline{(5 + 9i) + (2 - 4i)}$

$= \overline{5 + 9i} + \overline{2 - 4i} = 5 - 9i + 2 + 4i = 7 - 5i$

(ii) $\frac{10-5i}{6+2i} = \frac{10-5i}{6+2i} \times \frac{6-2i}{6-2i} = \frac{60-20i-30i+10i^2}{6^2+2^2}$

$= \frac{60-50i-10}{36+4} = \frac{50-50i}{40} = \frac{10(5-5i)}{40} = \frac{5}{4} - \frac{5i}{4} = \frac{5(1-i)}{4}$

(iii) $3\bar{i} + \frac{1}{2-i} = - 3i + \frac{1}{2-i} \times \frac{2+i}{2+i} = - 3i + \frac{2+i}{2^2+1^2}$

$= - 3i + \frac{2+i}{5} = \frac{-15i+2+i}{5} = \frac{2-14i}{5} = \frac{2}{5}(1 - 7i)$

---

(2) Find the rectangular form of the following $\quad z = x + iy$.

(i) $\mathrm{Re}\left(\frac{1}{z}\right)$ $\quad z = x + iy$

$\frac{1}{z} = z^{-1} = \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2}$ $\quad \therefore \mathrm{Re}\left(\frac{1}{z}\right) = \frac{x}{x^2+y^2}$

(ii) $\mathrm{Re}\,(i\bar{z})$

$z = x + iy$ $\quad \therefore \bar{z} = x - iy$

$i\bar{z} = i(x - iy) = ix - i^2 y = y + ix$ $\quad \therefore \mathrm{Re}\,(i\bar{z}) = y$

(iii) $\mathbf{Im\ (3z + 4\bar{z} - 4i)}$

$3z + 4\bar{z} - 4i = 3(x + iy) + 4(x - iy) - 4i$

$= 3x + i3y + 4x - i4y - 4i$

$= (3x + 4x) + i(3y - 4y - 4)$

$= 7x + i(-y - 4)$

$Im(3z + 4\bar{z} - 4i) = -y - 4$

---

(3) If $\mathbf{z_1 = 2 - i}$, $\mathbf{z_3 = -4 + 3i}$, Find the inverses of $\mathbf{z_1 z_2}$ & $\mathbf{\frac{z_1}{z_2}}$.

Solution :

$z_1 z_2 = (2 - i)(-4 + 3i) = -8 + 6i + 4i - 3i^2$

$= -8 + 10i + 3 = -5 + 10i$

$(z_1 z_2)^{-1} = \frac{-5}{(-5)^2+10^2} + i\frac{-10}{(-5)^2+10^2} = \frac{-5-10i}{25+100} = \frac{-5(1+2i)}{125}$

$= \frac{1}{25}(-1 - 2i)$

$\left(\frac{z_1}{z_2}\right)^{-1} = \frac{z_2}{z_1} = \frac{-4+3i}{2-i} \times \frac{2+i}{2+1} = \frac{-8-4i+6i+3i^2}{2^2+1^2}$

$= \frac{-8+2i-3}{4+1} = \frac{1}{5}(-11 + 2i)$

EXERCISE 2.5

1. (i) $\left|\frac{2i}{3+4i}\right| = \frac{|2i|}{|3+4i|} = \frac{|2||i|}{\sqrt{3^2+4^2}} = \frac{2(1)}{\sqrt{25}} = \frac{2}{5}$

---

1 (ii) $\left|\frac{2-i}{1+i} + \frac{1-2i}{1-i}\right| = \left|\frac{(2-i)(1-i)+(1-2i)(1+i)}{(1+i)(1-i)}\right|$

$= \left|\frac{2-2i-i+i^2+1+i-2i-2i^2}{1^2+1^2}\right|$

$= \left|\frac{2-3i-1+1-i+2}{1+1}\right| = \left|\frac{4-4i}{2}\right| = \frac{\sqrt{4^2+(-4)^2}}{2}$

$= \frac{\sqrt{32}}{2} = \frac{4\sqrt{2}}{2} = 2\sqrt{2}$

EXERCISE 2.6

(3) Obtain cartesian form of the locus of $= x + iy$

(i) $[Re\ (iz)]^2 = 3$

Solution :

$\mathbf{z = x + iy}$

$iz = i(x + iy) = ix + i^2y = -y + ix$

$Re\ (iz) = -y$

$[Re(iz)]^2 = (-y)^2 = y^2$

$\therefore \mathbf{[Re(iz)]^2 = 3 \Rightarrow y^2 = 3}$

(ii) $Im\ [(1 - i)z + 1] = 0$.

Soln: $z = x + iy$

$(1 - i)z + 1 = (1 - i)(x + iy) + 1$

$= x + iy - ix - i^2y + 1$

$= x + iy - ix + y + 1$

$= (x + y + 1) + i(y - x)$

$\mathbf{Im\ [(1 - i)z + 1] = 0 \Rightarrow y - x = 0 \Rightarrow x = y}$

(iii) $|z + i| = |z - 1|$ $[z = x + iy]$

$|x + iy + i| = |x + iy - 1|$

$|x + i(y + 1)| = |x - 1 + iy|$

$\Rightarrow \sqrt{x^2 + (y + 1)^2} = \sqrt{(x - 1)^2 + y^2}$

$\Rightarrow x^2 + (y + 1)^2 = (x - 1)^2 + y^2$

$\Rightarrow x^2 + y^2 + 2y + 1 = x^2 - 2x + 1 + y^2 \Rightarrow 2x + 2y = 0$

$\Rightarrow x + y = 0$ **Locus of z is $x + y = 0$**

(iv) $\bar{z} = z^{-1} = \frac{1}{z}$

$\Rightarrow \mathbf{z\bar{z} = 1 \Rightarrow |z|^2 = 1 \Rightarrow |x + iy|^2 = 1 \Rightarrow \sqrt{x^2 + y^2} = 1}$

$\Rightarrow \mathbf{x^2 + y^2 = 1}$

---

(4) show that the following eqns represent a circle , and find its centre and radius . (each 2 Mark)

(i) $|z - 2 - i| = 3 \Rightarrow |z - (2 + i)| = 3$

It is in the form of $|z - z_0| = a$; it forms or rep eqn of circle

$\mathbf{z_0 = 2 + i}$ i.e $(2, 1)$ $a = 3$

(ii) $|2z + 2 - 4i| = 2$

$\div 2$ $\mathbf{|z + 1 - 2i| = 1 \Rightarrow |z - (-1 + 2i)| = 1}$

It is in the form of $|z - z_0| = a$; it forms or rep eqn of circle

$\mathbf{z_0 = -1 + 2i}$ i.e. $(-1, 2)$ $a = 1$

(ii) $|3z - 6 + 12i| = 8$

$\div 3$ $|z - 2 + 4i| = \frac{8}{3}$ $\Rightarrow |z - (+2 - 4i) = \frac{8}{3}$

It is in the form of $|z - z_0| = a$; it forms or rep eqn of circle

center $z_0 = 2 - 4i$ i.e $(2, -4)$ $a = \frac{8}{3}$

---

5. Obtain the cartesian eqn for the locus of

$z = x + iy$ in each of the following cases.

(i) $|z - 4| = 16$ $z = x + iy$

$\mathbf{|x + iy - 4| = 16 \Rightarrow |x - 4 + iy| = 16}$

$\mathbf{\sqrt{(x - 4)^2 + y^2} = 16}$

$\mathbf{x^2 - 8x + 16 + y^2 = 16^2 = 256}$

$\mathbf{x^2 + y^2 - 8x + 16 - 256 = 0 \Rightarrow x^2 + y^2 - 8x - 240 = 0}$

(ii) $|z - 4|^2 - |z - 1|^2 = 16$ Given $z = x + iy$

$\mathbf{|x + iy - 4|^2 - |x + iy - 1|^2 = 16}$

$\mathbf{|(x - 4) + iy|^2 - |(x - 1) + iy|^2 = 16}$

$\mathbf{\left[\sqrt{(x - 4)^2 + y^2}\right]^2 - \left[\sqrt{(x - 1)^2 + y^2}\right]^2 = 16}$

$\mathbf{(x - 4)^2 + y^2 - [(x - 1)^2 + y^2] = 16}$

$\mathbf{x^2 - 8x + 16 + y^2 - (x^2 - 2x + 1 + y^2) = 16}$

$\mathbf{x^2 - 8x + 16 + y^2 - x^2 + 2x - 1 - y^2 = 16}$

$\mathbf{-8x + 16 + 2x - 1 - 16 = 0}$

$\mathbf{-6x - 1 = 0 \Rightarrow -6x = 1}$

Locus of z ix $x = \frac{-1}{6}$ **or $6x + 1 = 0$**

EXERCISE 2.7

1. Write the polar form.

(i) $\mathbf{2 + i\,2\sqrt{3} = r(\cos\theta + i\sin\theta)}$ $\quad \mathbf{a = 2} \quad \mathbf{b = 2\sqrt{3}}$

$r = \sqrt{a^2+b^2} = \sqrt{2^2 + (2\sqrt{3})^2} = \sqrt{4+12} = \sqrt{16} = 4$

$\alpha = \tan^{-1}\left|\frac{b}{a}\right| = \tan^{-1}\left|\frac{2\sqrt{3}}{2}\right| = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$

$2 + 2\sqrt{3}\,\text{I}$ lies in I quadrant $\therefore \theta = \alpha = \frac{\pi}{3}$

$\therefore 2 + i2\sqrt{3} = 4\left[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right]$

General form

$\mathbf{2 + i2\sqrt{3} = 4\left[\cos\left(2k\pi + \frac{\pi}{3}\right) + i\sin\left(2k\pi + \frac{\pi}{3}\right)\right] \quad k \in z}$

(ii) $\mathbf{3 - i\sqrt{3} = r(\cos\theta + i\sin\theta)}$ $\quad \mathbf{a = 3} \quad \mathbf{b = -\sqrt{3}}$

$r = \sqrt{a^2+b^2} = \sqrt{3^2 + (-\sqrt{3})^2} = \sqrt{9+3} = \sqrt{12} = 2\sqrt{3}$

$\alpha = \tan^{-1}\left|\frac{b}{a}\right| = \tan^{-1}\left|-\frac{\sqrt{3}}{3}\right| = \tan^{-1}\left|-\frac{\sqrt{3}}{\sqrt{3}\sqrt{3}}\right|$

$= \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

$\theta = -\alpha = -\frac{\pi}{6}$ $\left(\begin{matrix}3 - i\sqrt{3} \text{ lies in} \\ \text{IV quadrant}\end{matrix}\right)$

$3 - i\sqrt{3} = 2\sqrt{3}\left[\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right]$

$= 2\sqrt{3}\left[\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right]$

$\mathbf{3 - i\sqrt{3} = 2\sqrt{3}\left[\cos\left(2k\pi + \frac{\pi}{6}\right) - i\sin\left(2k\pi + \frac{\pi}{6}\right)\right] \; k \in z}$

(ii) $\mathbf{-2 - i2 = r(\cos\theta + i\sin\theta)}$

$a = -2 \quad b = -2$

$r = \sqrt{(-2)^2 + (-2)^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$

$\alpha = \tan^{-1}\left|\frac{b}{a}\right| = \tan\left|\frac{-2}{-2}\right| = \tan^{-1}(1) = \frac{\pi}{4}$

$\theta = -\pi + \alpha = -\pi + \frac{\pi}{4}$ $\quad$ ($-2 - i2$ lies in III quadrant)

$= \frac{-4\pi+\pi}{4} = \frac{-3\pi}{4}$

$-2 - i2 = 2\sqrt{2}\left[\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right]$

$\mathbf{-2 - i2 = 2\sqrt{2}\left[\cos\left(2k\pi - \frac{3\pi}{4}\right) + i\sin\left(2k\pi - \frac{3\pi}{4}\right)\right] \;, k \in z}$

EXERCISE 2.8

1. If $\mathbf{\omega \neq 1}$ **; S.T.** $\mathbf{\frac{a+b\omega+c\omega^2}{b+c\omega+a\omega^2} + \frac{a+b\omega+c\omega^2}{c+a\omega+b\omega^2} = -1}$

L.H.S= $\mathbf{\frac{a+b\omega+c\omega^2}{b+c\omega+a\omega^2} \times \frac{\omega}{\omega} + \frac{a+b\omega+c\omega^2}{c+a\omega+b\omega^2} \times \frac{\omega^2}{\omega^2}}$

$\mathbf{= \frac{(a+b\omega+c\omega^2)\omega}{b\omega+c\omega^2+a\omega^3} + \frac{a+b\omega+c\omega^2}{c\omega^2+a\omega^3+b\omega^2} \times \omega^2}$

$\mathbf{= \frac{(a+b\omega+c\omega^2)\omega}{a+b\omega+c\omega^2} + \frac{a+b\omega+c\omega^2}{c\omega^2+a+b\omega} \times \omega^2 \quad = \omega + \omega^2 = -1}$

3 MARKS

EXERCISE 2.2

(2) Given the complex number $\mathbf{z = 2 + 3i}$**, represent the complex number in Argand diagram**

(i) $z, iz, z + iz$

$z = 2 + 3i$

$iz = i(2+3i) = 2i + 3i^2 = 2i + 3(-1) = -3 + 2i$

$z + iz = 2 + 3i - 3 + 2i = -1 + 5i$

(ii) $z = 2 + 3i \qquad z, -iz, z - iz$

$z = 2 + 3i$

$-iz = -i(2+3i) = -2i - 3i^2 = -2i + 3 = 3 - 2i$

$z - iz = z + (-iz) = 2 + 3i + 3 - 2i = 5 + i$

(3) **Find x and y**

$\mathbf{(3-i)x - (2-i)y + 2i + 5 = 2x + (-1+2i)y + 3 + 2i}$

$3x - ix - 2y + yi + 2i + 5 = 2x - y + i2y + 3 + 2i$

$(3x - 2y + 5) + i(-x + y + 2) = (2x - y + 3) + i(2y + 2)$

Eqn Real $3x - 2y + 5 = 2x - y + 3$

$3x - 2y - 2x + y = 3 - 5$

$x - y = -2 \qquad$ - (1)

Eqn Img $-x + y + 2 = 2y + 2$

$-x + y - 2y = 2 - 2$

$-x - y = 0 \qquad$ - (2)

$x - y = -2$

$-x - y = 0$

$-2y = -2$

$y = 1$

$y = 1 \qquad -x - 1 = 0$

$-x = 1$

$x = -1, \quad y = 1$

EXERCISE 2.3

1. $\mathbf{z_1 = 1 - 3i, \; z_2 = -4i, \; z_3 = 5}$

(i) **S.T.** $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$

L.H.S $= (z_1 + z_2) + z_3$

$= [1 - 3i + (-4i)] + 5 = (1 - 3i - 4i) + 5$

$= 1 - 7i + 5 = 6 - 7i \qquad$ - (1)

R.H.S $= z_1 + (z_2 + z_3) \quad = 1 - 3i + (-4i + 5)$

$= 1 - 3i - 4i + 5 \quad = 6 - 7i \qquad$ - (2)

(1) = (2)

LHS = RHS

$\therefore \mathbf{(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)}$

(ii) $\mathbf{(z_1z_2)z_3 = z_1(z_2z_3)}$

L.H.S : $z_1z_2 = (1-3i)(-4i) = -4i + 12i^2 = -4i - 12$

$(z_1z_2)z_3 = (-12-4i)5 = -60 - 20i$ - (3)

R.H.S : $z_2z_3 = (-4i)5 = -20i$

$z_1(z_2z_3) = (1-3i)(-20i) = -20i + 60i^2$

$= -60 - 20i$ - (4)

(3) =(4) $\mathbf{(z_1z_2)z_3 = z_1(z_2z_3)}$

---

(2) $\mathbf{z_1 = 3 \quad z_2 = -7i \quad z_3 = 5 + 4i}$

(i) $\mathbf{z_1(z_2 + z_3) = z_1z_2 + z_1z_3}$

L.H.S : $z_2 + z_3 = -7i + 5 + 4i = 5 - 3i$

$z_1(z_2 + z_3) = 3(5-3i) = 15 - 9i$ - (1)

R.H.S : $z_1z_2 = 3(-7i) = -21i$

$z_1z_3 = 3(5+4i) = 15 + 12i$

$z_1z_2 + z_1z_3 = -21i + 15 + 12i$

$= 15 - 9i$ - (2)

(1) = (2) $\mathbf{z_1(z_2 + z_3) = z_1z_2 + z_1z_3}$

(ii) $\mathbf{(z_1 + z_2)z_3 = z_1z_3 + z_2z_3}$

L.H.S : $z_1 + z_2 = 3 + (-7i) = 3 - 7i$

$(z_1 + z_2)z_3 = (3-7i)(5+4i)$

$= 15 + 12i - 35i - 28i^2$

$= 15 - 23i + 28 = 43 - 23i$ - (3)

R.H.S : $z_1z_3 = 3(5+4i) = 15 + 12i$

$z_2z_3 = -7i(5+4i) = -35i - 28i^2 = 28 - 35i$

$z_1z_3 + z_2z_3 = 15 + 12i + 28 - 35i$

$= 43 - 23i$ - (4)

(3) = (4) $\mathbf{(z_1 + z_2)z_3 = z_1z_3 + z_2z_3}$

---

(3) Find the additive & multiplicative inverse of following complex numbers.

$\mathbf{z = a + ib} \qquad \mathbf{z^{-1} = \frac{a}{a^2+b^2} + i\frac{-b}{a^2+b^2}}$

(i) $\mathbf{z_1 = 2 + 5i}$ a = 2 & b = 5

additive inverse $-z_1 = -2 - 5i$

multiplicative inverse

$z_1^{-1} = \frac{2}{2^2+5^2} + i\frac{-5}{2^2+5^2} = \mathbf{\frac{2}{29} - \frac{5i}{29}}$

(ii) $\mathbf{z_2 = -3 - 4i}$ a = -3 & b = -4

additive inverse: $-z_2 = -(-3-4i) = 3 + 4i$

Multiplicative inverse :

$z_1^{-1} = \frac{-3}{(-3)^2+(-4)^2} + i\frac{-(-4)}{(-3)^2+(-4)^2} = \mathbf{\frac{-3}{25} + \frac{4\,i}{25}}$

(ii) $\mathbf{z_3 = 1 + i}$ a = 1 & b = 1

$-z_3 = -(1+i) = -1 - i$

$z_3^{-1} = \frac{1}{1^2+1^2} + i\frac{(-1)}{1^2+1^2} = \mathbf{\frac{1}{2} - \frac{i}{2}}$

EXERCISE 2.4

4. $\mathbf{u} = ?$ $\quad \mathbf{v = 3 - 4i} \quad \mathbf{w = 4 + 3i}$ & $\mathbf{\frac{1}{u} = \frac{1}{v} + \frac{1}{w}}$

$\frac{1}{u} = \frac{1}{3-4i} \times \frac{3+4i}{3+4i} = \frac{3+4i}{3^2+4^2} = \frac{3+4i}{25}$

$\frac{1}{w} = \frac{1}{4+3i} \times \frac{4-3i}{4-3i} = \frac{4-3i}{4^2+3^2} = \frac{4-3i}{25}$

$\frac{1}{u} = \frac{1}{v} + \frac{1}{w} = \frac{3+4i}{25} + \frac{4-3i}{25} = \frac{7+i}{25}$

$\frac{1}{u} = \frac{7+i}{25}$

$u = \frac{1}{\frac{7+i}{25}} = \frac{25}{7+i} \times \frac{7-i}{7-i} = \frac{25(7-i)}{7^2+1^2}$

$= \frac{25(7-i)}{50} = \frac{1}{2}(7 - i)$

---

5. $\mathbf{z = \bar{z}} \qquad \mathbf{z = a + ib}$

$\Leftrightarrow x + iy = \overline{x + iy} \qquad \bar{z} = a - ib$

$\Leftrightarrow x + iy = x - iy \qquad z + \bar{z} = 2a$

$\Leftrightarrow 2iy = x - x \qquad z + \bar{z} = 2Re(z)$

$\Leftrightarrow 2iy = 0 \qquad Re\,(z) = \frac{z+\bar{z}}{2}$

$\Leftrightarrow y = 0 \qquad z - \bar{z} = 2ib$

$\Leftrightarrow z = x \qquad z - \bar{z} = 2i\ im(z)$

$\Leftrightarrow$ **z is purely real** $\qquad im\,(z) = \frac{z-\bar{z}}{2i}$

---

6. Find the least tive integer n such that

$\mathbf{(\sqrt{3} + i)^n}$ **(i) real (i) Imaginary**

(i) $(\sqrt{3} + i)^1 = \sqrt{3} + i$ Complex no.

(ii) $(\sqrt{3} + i)^2 = (\sqrt{3})^2 + 2\sqrt{3}i + i^2$

$= 3 + 2\sqrt{3}i - 1 = 2 + 2\sqrt{3}i$

(iii) $(\sqrt{3} + i)^3 = (\sqrt{3} + i)^2(\sqrt{3} + i)$

$= (2 + 2\sqrt{3}i)(\sqrt{3} + i)$

$= 2\sqrt{3} + 2i + 6i + 2\sqrt{3}i^2$

$= 2\sqrt{3} + 8i - 2\sqrt{3}$

$= 8i$ purely imaginary

(iv) $(\sqrt{3} + i)^4 = (\sqrt{3} + i)^3(\sqrt{3} + i) = 8i(\sqrt{3} + i)$

$= 8\sqrt{3}i + 8i^2 = -8 + 8\sqrt{3}i$

(v) $(\sqrt{3} + i)^5 = (\sqrt{3} + i)^3(\sqrt{3} + i)^2$

$= 8i(2 + 2\sqrt{3}i) = 16i + 16\sqrt{3}i^2$

(vi) $(\sqrt{3} + i)^6 = (\sqrt{3} + i)^3(\sqrt{3} + i)^3 = 8i(8i) = 64i^2$

$= -64$ purely real

$n = 6 \quad (\sqrt{3} + i)^n$ is real

$n = 3 \quad (\sqrt{3} + i)^n$ is imaginary

7. (i) Show that $(2+i\sqrt{3})^{10}-(2-i\sqrt{3})^{10}$ purely imaginary

Solution :

$z=(2+i\sqrt{3})^{10}-(2-i\sqrt{3})^{10}$

$\bar{z}=\overline{(2+i\sqrt{3})^{10}-(2-i\sqrt{3})^{10}}$

$=\overline{(2+i\sqrt{3})^{10}}-\overline{(2-i\sqrt{3})^{10}}$

$=\left(\overline{2+i\sqrt{3}}\right)^{10}-\left(\overline{2-i\sqrt{3}}\right)^{10}$

$=\left(2-i\sqrt{3}\right)^{10}-\left(2+i\sqrt{3}\right)^{10}$

$=-\left[\left(2+i\sqrt{3}\right)^{10}-\left(2-i\sqrt{3}\right)^{10}\right]$

$\bar{z}=-z$ ∴ **z is purely imaginary .**

EXERCISE 2.5

(2) For any two complex numbers z_1 and z_2 such that $|z_1|=|z_2|=1$ $z_1z_2\neq-1$ then show that $\frac{z_1+z_2}{1+z_1z_2}$ is a real number.

Solution :

$|z_1|=1$ $|z_2|=1$

$|z_1|^2=1$ $|z_2|=1$

$z_1\bar{z}_1=1$ $z_2\bar{z}_2=1$

$z_1=\frac{1}{\bar{z}_1}$ $z_2=\frac{1}{\bar{z}_2}$

let $z=\frac{z_1+z_2}{1+z_1z_2}=\frac{\frac{1}{\bar{z}_1}+\frac{1}{\bar{z}_2}}{1+\frac{1}{\bar{z}_1}\cdot\frac{1}{\bar{z}_2}}=\frac{\frac{\bar{z}_1+\bar{z}_2}{\bar{z}_1\bar{z}_2}}{\frac{\bar{z}_1\bar{z}_2+1}{\bar{z}_1\bar{z}_2}}=\frac{\bar{z}_1+\bar{z}_2}{1+\bar{z}_1\bar{z}_2}=\overline{\frac{z_1+z_2}{1+z_1z_2}}$

$=\overline{\left(\frac{z+z_2}{1+z_1z_2}\right)}$

$z=\bar{z}$ therefore z is purely real .

---

3. Which one of the point $10-8i$, $11+6i$ is closest to $1+i$.

Solution : A , B , C rep c'x numbers

$z_1=10-8i$, $z_2=11+6i$, $z_3=1+i$

$AC=|z_1-z_3|=|10-8i-1-i|=|9-9i|$

$=\sqrt{9^2+(-9)^2}=\sqrt{81+81}=\sqrt{162}$

$BC=|z_2-z_3|=|11+6i-1-i|=|10+5i|$

$=\sqrt{10^2+5^2}=\sqrt{100+25}=\sqrt{125}$

$\sqrt{125}<\sqrt{162}$ ∴ **$11+6i$ is closer to $1+i$**

---

(4) If $|z|=3$, Show $7\leqslant|z+6-8i|<13$.

Let $z_1=6-8i$

$|z_1|=\sqrt{6^2+(-8)^2}=\sqrt{36+64}=\sqrt{100}=10$

Wehave $||z|-|z_1||\leq|z+z_1|\leq|z|+|z_1|$

$|3-10|\leq|z+6-8i|\leq3+10.$

$|-7|\leq|z+6-8i|\leq13$

$\Rightarrow 7\leq|z+6-8i|\leq13.$

---

(5) If $|z|=1$ show that $2\leq|z^2-3|<4$

Solution :

let $z_1=-3$ ∴ $|z_1|=|-3|=3$.

$|z|=1$ $\Rightarrow|z^2|=|z|^2=1^2=1.$

we know $||z|^2-|z_1||\leq|z^2+(-3)|\leq|z|^2+|z_1|$

$|1^2-3|\leq|z^2-3|\leq1+3$

$|-2|\leq|z^2-3|\leq4$

$2\leq|z^2-3|\leq4$

---

(8) The area of triangle formed by the vertices z , iz , $z+iz$ is 50 sq.units. find $|z|$.

Solution :

Let A , B , C represent C'x nos z , iz , z + iz respectively.

$AB=|z-iz|=|z(1-i)|=|z||1-i|$

$=|z|\sqrt{1^2+(-1)^2}=|z|\sqrt{1+1}=\sqrt{z}|2|$

$BC=|iz-z-iz|=|-z|=|z|$

$AC=|z-(z+iz)|=|z-z-iz|=|-iz|$

$=|-i||z|=|z|$

AC = BC isosceles right triangle.

$AC^2+BC^2=|z|^2+|z|^2=2|z|^2=AB^2$

∴ ABC is an isosceles right triangle .

Area $=\frac{1}{2}$BC AC $=50$

$|z||z|=100$

$|z|^2=100$

$|z|=10$ $|z|=-10$ not possible

---

(9) S.T $z^3+2\bar{z}=0$ has five solution .

Solution :

$z^3+2\bar{z}=0$ - (1)

$z^3=-2\bar{z}$

$|z|^3=|-2||\bar{z}|$

$|z|^3=2|z|$

$|z|^3-2|z|=0$

$|z|(|z|^2-2)=0$

$|z|=0$ $|z|^2-2=0$

$z=0$ $|z|^2=2\Rightarrow z\bar{z}=2$

$\bar{z}=\frac{2}{z}$

sub in (1) $z^3+2\cdot\frac{2}{z}=0\Rightarrow z^4+4=0$

$|z|=0$ $z^4+4=0$

$\Rightarrow z=0$ $z^4+4=0$ gives 4 solution

∴ It has five solution.

$\mathbf{z = a + ib.}$

$\mathbf{|z| = \sqrt{a^2 + b^2}}$

$\mathbf{\sqrt{a + ib} = \pm\left(\sqrt{\frac{|z|+a}{2}} + i\frac{b}{|b|}\sqrt{\frac{|z|-a}{2}}\right)}$

10 (i) Find the square root of $4 + 3i$

$z = 4 + 3i \quad a = 4 \qquad b = 3$

$|z| = \sqrt{4^2 + 3^2} = \sqrt{16 + 9} = \sqrt{25} = 5$

$\sqrt{a + ib} = \pm\left(\sqrt{\frac{|z|+a}{2}} + \frac{ib}{|b|}\sqrt{\frac{|z|-a}{2}}\right)$

$\sqrt{4 + 3i} = \pm\left[\sqrt{\frac{5+4}{2}} \pm i\frac{3}{|3|}\sqrt{\frac{5-4}{2}}\right] = \pm\left(\frac{\sqrt{9}}{\sqrt{2}} \pm i\frac{3}{3}\frac{\sqrt{1}}{\sqrt{2}}\right)$

$= \pm\left(\frac{\sqrt{9}}{\sqrt{2}} \pm \frac{\sqrt{1}}{\sqrt{2}}i\right) = \pm\left(\frac{3}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)$

(ii) Find the square root of $-6 + 8i$

$z = -6 + 8i \quad a = -6 \qquad b = 8$

$|z| = \sqrt{(-6)^2 + 8^2} = \sqrt{36 + 64} = \sqrt{100} = 10$

$\sqrt{a + ib} = \pm\left(\sqrt{\frac{|z|+a}{2}} + \frac{ib}{|b|}\sqrt{\frac{|z|-a}{2}}\right)$

$\sqrt{-6 + 8i} = \pm\left(\sqrt{\frac{10+(-6)}{2}} + i\frac{8}{|8|}\sqrt{\frac{10-(-6)}{2}}\right)$

$= \pm\left(\sqrt{\frac{10-6}{2}} \pm i\frac{8}{8}\sqrt{\frac{10+6}{2}}\right) = \pm\left(\sqrt{\frac{4}{2}} + i\sqrt{\frac{16}{2}}\right)$

$= \pm(\sqrt{2} + i\sqrt{8}) = \pm(\sqrt{2} + i2\sqrt{2})$

(iii) Find the square root of $-5 - 12i$

$z = -5 - 12i \qquad a = -5 \qquad b = -12$

$|z| = \sqrt{(-5)^2 + (-12)^2} = \sqrt{25 + 144} = \sqrt{169} = 13$

$\sqrt{a + ib} = \pm\left(\sqrt{\frac{|z|+a}{2}} + i\frac{b}{|b|}\sqrt{\frac{|z|-a}{2}}\right)$

$= \pm\left(\sqrt{\frac{13+(-5)}{2}} + i\left(\frac{-12}{12}\right)\sqrt{\frac{13-(-5)}{2}}\right) = \pm\left(\sqrt{\frac{8}{2}} + \left(\frac{-12}{12}\right)i\sqrt{\frac{18}{2}}\right)$

$= \pm(\sqrt{4} - i\sqrt{9}) = \pm(2 - i3) = \pm(2 - 3i)$

__Note :__

$\mathbf{z = a + ib}$

$\mathbf{|z| = \sqrt{a^2 + b^2}}$

$\mathbf{(a + b)^2 = a^2 + 2ab + b^2}$

$\mathbf{(a - b)^2 = a^2 - 2ab + b^2}$

__EXERCISE 2.6__

(1) $\mathbf{z = x + iy}$ is a complex number such that $\left|\frac{\mathbf{z-4i}}{\mathbf{z+4i}}\right| = \mathbf{1}$. Show that locus of z is real axis .

Solution : $\quad \mathbf{z = x + iy}$

Given: $\left|\frac{\mathbf{z-4i}}{\mathbf{z+4i}}\right| = \mathbf{1}$

$\frac{|z-4i|}{|z+4i|} = 1$

$|z - 4i| = |z + 4i|$

$|x + iy - 4i| = |x + iy + 4i|$

$|x + i(y - 4)| = |x + i(y + 4)|$

$\sqrt{x^2 + (y - 4)^2} = \sqrt{x^2 + (y + 4)^2}$

$x^2 + (y - 4)^2 = x^2 + (y + 4)^2$

$x^2 + y^2 - 8y + 16 = x^2 + y^2 + 8y + 16$

$\Rightarrow -16y = 0$

$\mathbf{y = 0}$ **equation of x – axis**

__EXERCISE 2.7__

1. (iv) $\mathbf{\frac{i-1}{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}}}$

Consider $\mathbf{i - 1 = -1 + i = r(\cos\theta + i\sin\theta)}$

$a = -1 \qquad b = 1$

$r = \sqrt{a^2 + b^2} = \sqrt{(-1)^2 + 1^2} = \sqrt{1 + 1} = \sqrt{2}$

$\alpha = \tan^{-1}\left|\frac{b}{a}\right| = \tan^{-1}\left|\frac{1}{-1}\right| = \tan^{-1}(1) = \frac{\pi}{4}$

$\theta = \pi - \alpha = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$ $(-1 + i$ lies in II quadrant$)$

$i - 1 = -1 + i = \sqrt{2}\left[\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right]$

$\frac{i-1}{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}} = \frac{\sqrt{2}\left[\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right]}{\cos\frac{\pi}{3} + i\sin\frac{3\pi}{4}}$

$= \sqrt{2}\left[\cos\left(\frac{3\pi}{4} - \frac{\pi}{3}\right) + i\sin\left(\frac{3\pi}{4} - \frac{\pi}{3}\right)\right]$

$= \sqrt{2}\left[\cos\frac{9\pi-4\pi}{12} + i\sin\frac{9\pi-4\pi}{12}\right]$

$= \sqrt{2}\left[\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right]$

$\mathbf{\frac{i-1}{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}} = \sqrt{2}\left[\cos\left(2k\pi + \frac{5\pi}{12}\right) + i\sin\left(2k\pi + \frac{5\pi}{12}\right)\right]}$

---

(2) Find the rectangular form.

(i) $\mathbf{\left[\cos\left(\frac{\pi}{6}\right) + i\sin\frac{\pi}{6}\right]\left[\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right]}$

$= \cos\left(\frac{\pi}{6} + \frac{\pi}{12}\right) + i\sin\left(\frac{\pi}{6} + \frac{\pi}{12}\right)$

$= \cos\left(\frac{2\pi+\pi}{12}\right) + i\sin\left(\frac{2\pi+\pi}{12}\right) = \cos\left(\frac{3\pi}{12}\right) + i\sin\left(\frac{3\pi}{12}\right)$

$\mathbf{= \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + i\cdot\frac{1}{\sqrt{2}}}$

(ii) $\frac{\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}}{2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)} = \frac{\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)}{2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)}$

$=\frac{1}{2}\left[\cos\left(-\frac{\pi}{6}-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{6}-\frac{\pi}{3}\right)\right]$

$=\frac{1}{2}\left[\cos\left(\frac{-\pi-2\pi}{6}\right)+i\sin\left(\frac{-\pi-2\pi}{6}\right)\right]$

$=\frac{1}{2}\left[\cos\left(-\frac{3\pi}{6}\right)+i\sin\left(-\frac{3\pi}{6}\right)\right]=\frac{1}{2}\left[\cos\left(\frac{-\pi}{2}\right)+i\sin\left(-\frac{\pi}{2}\right)\right]$

$=\frac{1}{2}\left[\cos\frac{\pi}{2}-i\sin\frac{\pi}{2}\right]=\frac{1}{2}(0-i) \quad = \mathbf{\frac{1}{2}i}$

---

(3) If $(x_1+iy_1)(x_2+iy_2)\ldots(x_n+iy_n)=a+ib$
show that $(x_1^2+y_1^2)(x_2^2+y_2^2)\ldots(x_n^2+y_n^2)=a^2+b^2$

$\sum_{r=1}^{n}\tan^{-1}\left(\frac{y_r}{x_r}\right)=k\pi+\tan^{-1}\left(\frac{b}{a}\right) \qquad k\in z$

Solution :

(i) $(x_1+iy_1)(x_2+iy_2)\ldots(x_n+iy_n)=a+ib$

$|(x_1+iy_1)(x_2+iy_2)(x_3+iy_3)|=|a+ib|$

$|(x_1+iy_1)||(x_2+iy_2)|\ldots|(x_3+iy_3)|=|a+ib|$

$\sqrt{x_1^2+y_1^2}\sqrt{x_2^2+y_2^2}\cdots\sqrt{x_n^2+y_n^2}=\sqrt{a^2+b^2}$

On squaring

$(x_1^2+y_1^2)(x_2^2+y_2^2)\ldots(x_n^2+y_n^2)=a^2+b^2$

(ii) arg $[(x_1+iy_1)(x_2+iy_2)\ldots(x_n+iy_n)]=\arg(a+ib)$

$\arg(x_1+iy_1)+\arg(x_2+iy_2)+\cdots+\arg(x_n+iy_n)$

$=\arg(a+ib)$

$\Rightarrow \tan^{-1}\left(\frac{y_1}{x_1}\right)+\tan^{-1}\left(\frac{y_2}{x_2}\right)+\cdots+\tan^{-1}\left(\frac{y_n}{x_n}\right) = \tan^{-1}\left(\frac{b}{a}\right)$

∴ Given solution

$\tan^{-1}\left(\frac{y_1}{x_1}\right)+\tan^{-1}\left(\frac{y_2}{x_2}\right)+\cdots+\tan^{-1}\left(\frac{y_n}{x_n}\right)$

$= k\pi+\tan^{-1}\left(\frac{b}{a}\right)$

EXERCISE 2.8

2) Show that $\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^5+\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)^5=-\sqrt{3}$

Solution : $\frac{\sqrt{3}}{2}+\frac{i}{2}=r(\cos\theta+i\sin\theta)$

$r=\sqrt{a^2+b^2}=\sqrt{\left(\frac{\sqrt{3}}{2}\right)^2+\left(\frac{1}{2}\right)^2}=\sqrt{\frac{3}{4}+\frac{1}{4}}=1$

$\alpha=\tan^{-1}\left|\frac{b}{a}\right|=\tan^{-1}\left|\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right|=\tan^{-1}\left|\frac{1}{\sqrt{3}}\right|=\frac{\pi}{6}$

$\theta=\alpha=\frac{\pi}{6} \qquad \because \frac{\sqrt{3}}{2}+\frac{i}{2}$ lies in I Quad

$\frac{\sqrt{3}}{2}+\frac{i}{2}=1\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$

$\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^5=\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)^5 = \mathbf{\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}}$

Similarly $\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)^5=\cos\frac{5\pi}{6}-i\sin\frac{5\pi}{6}$

$\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^5+\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)^5=\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}+\cos\frac{5\pi}{6}-i\sin\frac{5\pi}{6}$

$\mathbf{=2\cos\frac{5\pi}{6}=2\cos 150^\circ}$

$\mathbf{=2\cos(180^\circ-30^\circ)=2[-\cos 30^\circ]=-2\cdot\frac{\sqrt{3}}{2}=-\sqrt{3}}$

5. Solve: $z^3+27=0$

$\mathbf{z^3=-27 \quad =27\times-1}$

$z=(27)^{1/3}(-1)^{1/3}$

$z=(27)^{\frac{1}{3}}[\cos\pi+i\sin\pi]^{\frac{1}{3}}$

$z=3[\cos(2k\pi+\pi)+i\sin(2k\pi+\pi)]^{\frac{1}{3}}$

$=3\left[\cos(2k+1)\frac{\pi}{3}+i\sin(2k+1)\frac{\pi}{3}\right] \quad K=0,1,2$

$=3\,\text{cis}\,(2k+1)\frac{\pi}{3}$

$\mathbf{k=0;\ \ z_1=3\,cis\,\frac{\pi}{3}}$

$\mathbf{k=1;\ \ z_2=3\,cis\,\frac{3\pi}{3}=3\,cis\pi=-3}$

$\mathbf{k=2;\ \ z_3=3\,cis\,\frac{7\pi}{3}}$

---

(5) $\omega\neq 1$ cube roots of unity. S.T. roots of eqn $(z-1)^3+8=0$ are $-1,+1-2\omega,1-2\omega^2$

Solution :

$(z-1)^3+8=0$

$(z-1)^3=-8$

$(1-z)^3=8=2^3$

$\left(\frac{1-z}{2}\right)^3=1$

$\frac{1-z}{2}=(1)^{1/3}$

$\frac{1-z}{2}=1 \quad \frac{1-z}{2}=\omega \quad \frac{1-z}{2}=\omega^2$

$1-z=2 \quad 1-z=2\omega \quad 1-z=2\omega^2$

$\Rightarrow z=-1 \quad z=1-2\omega \quad z=1-2\omega^2$

∴ roots are $-1,1-2\omega,1-2\omega^2$

---

(7) Find he value $\sum_{k=1}^{8}\left(\cos\frac{2k\pi}{9}+i\sin\frac{2k\pi}{9}\right)$

Solution :

Let $x=\cos\frac{2k\pi}{9}+i\sin\frac{2k\pi}{9}$

$k=1 \quad x_1=\cos\frac{2\pi}{9}+i\sin\frac{2\pi}{9}=\omega$

$k=2 \quad x_2=\cos\frac{4\pi}{9}+i\sin\frac{4\pi}{9}=\left(\cos\frac{2\pi}{9}+i\sin\frac{2\pi}{9}\right)^2=\omega^2$

$k=3 \quad x_3=\cos\frac{6\pi}{9}+i\sin\frac{6\pi}{9}=\left(\cos\frac{2\pi}{9}+i\sin\frac{2\pi}{9}\right)^3=\omega^3$

$k=4 \quad x_4=\cos\frac{8\pi}{9}+i\sin\frac{8\pi}{9}=\left(\cos\frac{2\pi}{9}+i\sin\frac{2\pi}{9}\right)^4=\omega^4$

$k=5 \quad x_5=\cos\frac{10\pi}{9}+i\sin\frac{10\pi}{9}=\left(\cos\frac{2\pi}{9}+i\sin\frac{2\pi}{9}\right)^5=\omega^5$

$k=6 \quad x_6=\cos\frac{12\pi}{9}+i\sin\frac{12\pi}{9}=\left(\cos\frac{2\pi}{9}+i\sin\frac{2\pi}{9}\right)^6=\omega^6$

$k=7 \quad x_7=\cos\frac{14\pi}{9}+i\sin\frac{14\pi}{9}=\left(\cos\frac{2\pi}{9}+i\sin\frac{2\pi}{9}\right)^7=\omega^7$

$k=8 \quad x_8=\cos\frac{16\pi}{4}+i\sin\frac{16\pi}{9}=\left(\cos\frac{2\pi}{9}+i\sin\frac{2\pi}{9}\right)^8=\omega^8$

$\mathbf{\sum_{k=1}^{8}\cos\frac{2k\pi}{9}+i\sin\frac{2k\pi}{9}}$

$\mathbf{=\omega+\omega^2+\omega^3+\omega^4+\omega^5+\omega^6+\omega^7+\omega^8=-1}$

$\mathbf{(\because 1+\omega+\omega^2+\omega^3+\omega^4+\omega^5+\omega^6+\omega^7+\omega^8=0)}$

(9) If $z = 2 - 2i$. Find the rotation of z by θ radians by counter clockwise direction.

$z = 2 - 2i = r(\cos\theta + i\sin\theta)$

$a = 2 \quad b = -2$

$r = \sqrt{a^2 + b^2} = \sqrt{2^2 + (-2)^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{3}$

$\alpha = \tan^{-1}\left|\frac{b}{a}\right| = \tan^{-1}\left|\frac{-2}{2}\right| = \tan^{-1}(1)$

$\alpha = \frac{\pi}{4} \Rightarrow \theta = -\alpha = -\frac{\pi}{4}$ lies in IV Quadrant

$\therefore z = 2 - 2i = 2\sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right] = 2\sqrt{2}e^{-i\frac{\pi}{4}}$

(i) rotated by $\frac{\pi}{3}$

$z_1 = 2\sqrt{2}e^{-i\frac{\pi}{4}} \cdot e^{i\frac{\pi}{3}} = 2\sqrt{2}e^{i(-\frac{\pi}{4}+\frac{\pi}{3})} = 2\sqrt{2}e^{i\frac{\pi}{12}}$

(ii) rotated by $\frac{2\pi}{3}$

$z_2 = 2\sqrt{2}e^{-i\frac{\pi}{4}} \cdot e^{i\frac{2\pi}{3}} = 2\sqrt{2}e^{i(-\frac{\pi}{4}+\frac{2\pi}{3})} = 2\sqrt{2}e^{i\frac{5\pi}{12}}$

(iii) rotated by $\frac{3\pi}{2}$

$z_3 = 2\sqrt{2}e^{-i\frac{\pi}{4}} \cdot e^{i\frac{3\pi}{2}} = 2\sqrt{2}e^{i(-\frac{\pi}{4}+\frac{3\pi}{2})} = 2\sqrt{2}e^{i\frac{5\pi}{4}}$

---

(8) $\omega \neq 1$, S.T.

(i) $(1 - \omega + \omega^2)^6 + (1 + \omega - \omega^2)^6 = 128$

$L \cdot H \cdot S = (1 - \omega + \omega^2)^6 + (1 + \omega - \omega^2)^6$

$= (1 + \omega^2 - \omega)^6 + (1 + \omega - \omega^2)^6$

$= (-\omega - \omega)^6 + (-\omega^2 - \omega^2)^6 = (-2\omega)^6 + (-2\omega^2)^6$

$= (-2)^6\omega^6 + (-2)^6(\omega^2)^6 = 64\omega^6 + 64\omega^{12}$

$= 64 + 64 = \mathbf{128}$

(ii) $(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)\ldots\left(1+\omega^{2^{11}}\right) = 1$

L.H.S

$(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)\ldots\left(1+\omega^{2^{11}}\right)$

$= (1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)(1+\omega^{16})$

$(1+\omega^{32})(1+\omega^{64})(1+\omega^{128})\ldots\left(1+\omega^{2^{11}}\right)$

$= (1+\omega)(1+\omega^2)(1+\omega)(1+\omega^2)(1+\omega)(1+\omega^2)(1+\omega)$

$(1+\omega^2) \quad (1+\omega)(1+\omega^2)(1+\omega)(1+\omega^2)$

$= [(1+\omega)(1+\omega^2)]^6 = [1 + \omega^2 + \omega + \omega^2]^6$

$= (0+1)^6 = 1^6 = \mathbf{1}$

---

(3) Find the value of $\left[\frac{1+\sin\frac{\pi}{10}+i\cos\frac{\pi}{10}}{1+\sin\frac{\pi}{10}-i\cos\frac{\pi}{10}}\right]$

Solution :

let $z = \sin\frac{\pi}{10} + i\cos\frac{\pi}{10}$

$\because |z| = 1 \Rightarrow z^{-1} = \bar{z} = \sin\frac{\pi}{10} - i\cos\frac{\pi}{10}$

$\therefore \left[\frac{1+\sin\frac{\pi}{10}+i\cos\frac{\pi}{10}}{1+\sin\frac{\pi}{10}-i\cos\frac{\pi}{10}}\right]^{10} = \left[\frac{1+z}{1+\frac{1}{z}}\right]^{10} = \left[\frac{1+z}{\frac{z+1}{z}}\right]^{10} = (z)^{10}$

$= \left[\sin\frac{\pi}{10} + i\cos\frac{\pi}{10}\right]^{10}$

$= i^{10}\left[\cos\frac{\pi}{10} - i\sin\frac{\pi}{10}\right]^{10}$

$= i^8 \cdot i^2\left[\cos\frac{\pi}{10} \times 10 - i\sin\frac{\pi}{10} \times 10\right]$

$= -1[\cos\pi - i\sin\pi] = -1(-1) = 1$

5 MARKS

EXAMPLE 2.8(ii)

PROVE : $\left(\frac{19+9i}{5-3i}\right)^{15} - \left(\frac{8+i}{1+2i}\right)^{15}$ is purely imaginary

Solution :

$\frac{19+9i}{5-3i} = \frac{19+9i}{5-3i} \times \frac{5+3i}{5+3i} = \frac{95+57i+45i+27i^2}{5^2+3^2} = \frac{95+102i-27}{25+9}$

$= \frac{68+102i}{34} = 34\frac{(2+3i)}{34} = 2 + 3i$

$\frac{8+i}{1+2i} = \frac{8+i}{1+2i} \times \frac{1-2i}{1-2i} = \frac{8-16i+i-2i^2}{1^2+2^2}$

$= \frac{8-15i+2}{1+4} = \frac{10-15i}{5} = \frac{5(2-3i)}{5} = 2 - 3i$

Let $z = \left(\frac{19+9i}{5-3i}\right)^{15} - \left(\frac{8+i}{1+2i}\right)^{15} = (2+3i)^{15} - (2-3i)^{15}$

$\bar{z} = \overline{(2+3i)^{15} - (2-3i)^{15}}$

$= \overline{(2+3i)^{15}} - \overline{(2-3i)^{15}}$

$= (\overline{2+3i})^{15} - (\overline{2-3i})^{15}$

$= (2-3i)^{15} - (2+3i)^{15}$

$= -\left[(2+3i)^{15} - (2-3i)^{15}\right] = -z$

$\Rightarrow z = -\bar{z} \therefore z$ is purely imaginary

EXERCISE 2.4 7 (ii)

PROVE $\left(\frac{19-7i}{9+i}\right)^{12} + \left(\frac{20-5i}{7-6i}\right)^{12}$ is real .

Solution :

$\frac{19-7i}{9+i} = \frac{19-7i}{9+i} \times \frac{9-i}{9-i} = \frac{171-19i-63i+7i^2}{9^2+1^2}$

$= \frac{171-82i-7}{81+1} = \frac{164-82i}{82}$

$= \frac{82(2-i)}{82} = 2 - i$

$\frac{20-5i}{7-6i} = \frac{20-5i}{7-6i} \times \frac{7+6i}{7+6i}$

$= \frac{140+120i-35i-30i^2}{7^2+6^2}$

$= \frac{140+85i+30}{49+36}$

$= \frac{170+85i}{85} = \frac{85(2+i)}{85} = 2 + i$

Let $z = \left(\frac{19-7i}{9+i}\right)^{12} + \left(\frac{20-5i}{7-6i}\right)^{12}$

$z = (2-i)^{12} + (2+i)^{12}$

$\bar{z} = \overline{(2-i)^{12} + (2+i)^{12}}$

$= \overline{(2-i)^{12}} + \overline{(2+i)^{12}}$

$= (\overline{2-i})^{12} + (\overline{2+i})^{12}$

$= (2+i)^{12} + (2-i)^{12} = z$

$\therefore \bar{z} = z$, z is real .

EXAMPLE 2.14

show that the points $\mathbf{1}$, $\mathbf{\frac{-1}{2} + i\frac{\sqrt{3}}{2}}$ and $\mathbf{\frac{-1}{2} + i\frac{\sqrt{3}}{2}}$ forms a equilateral triangle.

Solution :

Let A, B, C represent $\mathbf{z_1 = 1 \,;\, z_2 = \frac{-1}{2} + i\frac{\sqrt{3}}{2} \,;\, z_3 = \frac{-1}{2} - i\frac{\sqrt{3}}{2}}$

$$AB = |z_1 - z_2| = \left|1 - \left(\frac{-1}{2} + i\frac{\sqrt{3}}{2}\right)\right| = \left|1 + \frac{1}{2} - i\frac{\sqrt{3}}{2}\right|$$

$$= \left|\frac{3}{2} - i\frac{\sqrt{3}}{2}\right| = \sqrt{\left(\frac{3}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{9}{4} + \frac{3}{4}} = \sqrt{\frac{12}{4}} = \sqrt{3}$$

$$BC = |z_2 - z_3| = \left|\frac{-1}{2} + i\frac{\sqrt{3}}{2} - \left(\frac{-1}{2} - i\frac{\sqrt{3}}{2}\right)\right|$$

$$= \left|\frac{-1}{2} + i\frac{\sqrt{3}}{2} + \frac{1}{2} + i\frac{\sqrt{3}}{2}\right| = \left|i\frac{2\sqrt{3}}{2}\right| = |i\sqrt{3}|$$

$$= \sqrt{0^2 + \sqrt{3}^2} = \sqrt{3}$$

$$AC = |z_1 - z_3| = \left|1 - \left(\frac{-1}{2} - i\frac{\sqrt{3}}{2}\right)\right| = \left|1 + \frac{1}{2} + i\frac{\sqrt{3}}{2}\right|$$

$$= \left|\frac{3}{2} + i\frac{\sqrt{3}}{2}\right| = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{9}{4} + \frac{3}{4}} = \sqrt{\frac{12}{4}} = \sqrt{3}$$

$\mathbf{AB = BC = AC}$ **. Therefore It forms equilateral triangle .**

EXAMPLE 2.15

$\mathbf{|z_1| = |z_2| = |z_3| = r \,,\, z_1 + z_2 + z_3 \neq 0;}$ **S.T** $\mathbf{\left|\frac{z_1z_2 + z_2z_3 + z_3z_1}{z_1 + z_2 + z_3}\right| = r}$

Solution: $|z_1| = r \Rightarrow |z_1|^2 = r^2 \Rightarrow z_1\overline{z_1} = r^2 \Rightarrow z_1 = \frac{r^2}{\overline{z_1}}$,

similarly $z_2 = \frac{r^2}{\overline{z_2}}$, $z_3 = \frac{r^2}{\overline{z_3}}$

$$z_1 + z_2 + z_3 = \frac{r^2}{\overline{z_1}} + \frac{r^2}{\overline{z_2}} + \frac{r^2}{\overline{z_3}} = r^2\left(\frac{1}{\overline{z_1}} + \frac{1}{\overline{z_2}} + \frac{1}{\overline{z_3}}\right)$$

$$= r^2\left(\frac{\bar{z}_2\bar{z}_3 + \bar{z}_1\bar{z}_3 + \bar{z}_1\bar{z}_2}{\bar{z}_1\bar{z}_1\bar{z}_3}\right) = r^2\frac{(\overline{z_2z_3} + \overline{z_1z_3} + \overline{z_1z_2})}{\overline{z_1z_2z_3}}$$

$$|z_1 + z_2 + z_3| = \frac{|r^2(\overline{z_2z_1 + z_1z_3 + z_1z_2})|}{|\overline{z_1z_2z_3}|} = |r^2|\frac{|z_2z_3 + z_1z_3 + z_1z_2|}{|z_1||z_2||z_3|}$$

$$|z_1 + z_2 + z_3| = r^2\frac{|z_2z_3 + z_1z_3 + z_1z_2|}{r.r.r}$$

$$\frac{r^3}{r^2} = \frac{|z_1z_2 + z_2z_3 + z_3z_1|}{|z_1 + z_2 + z_3|} \Rightarrow r = \frac{|z_1z_2 + z_2z_3 + z_3z_1|}{|z_1 + z_2 + z_3|}$$

---

(7) If $\mathbf{z_1}$, $\mathbf{z_2}$ and $\mathbf{z_3}$ are 3 complex nos, such that

$\mathbf{|z_1| = 1 \,, |z_2| = 2 \,, |z_3| = 3 \,,\, |z_1 + z_2 + z_3| = 1}$

Show that $\mathbf{|9z_1z_2 + 4z_1z_3 + z_2z_3| = 6}$.

Solution : $|z_1| = 1$ $\quad |z_2| = 2$ $\quad |z_3| = 3$

$|z_1|^2 = 1$ $\quad |z_2|^2 = 4$ $\quad |z_3|^2 = 9$

$z_1\overline{z_1} = 1$ $\quad z_2\overline{z_2} = 4$ $\quad z_3\overline{z_3} = 9$

$z_1 = \frac{1}{\overline{z_1}}$ $\quad z_2 = \frac{4}{\overline{z_2}}$ $\quad z_3 = \frac{9}{\overline{z_3}}$

$$\mathbf{|z_1 + z_2 + z_3| = 1 \Rightarrow \left|\frac{1}{\overline{z_1}} + \frac{4}{\overline{z_2}} + \frac{9}{\overline{z_3}}\right| = 1}$$

$$\mathbf{\left|\frac{\overline{z}_2\overline{z}_3 + 4\overline{z}_1\overline{z}_3 + 9\overline{z}_1\overline{z}_2}{\overline{z}_1\overline{z}_2\overline{z}_3}\right| = 1}$$

$$\mathbf{\frac{|\overline{z_2z_3} + 4\overline{z_1z_3} + 9\overline{z_1z_2}|}{|\overline{z_1z_2z_3}|} = 1}$$

$\mathbf{|\overline{z_2z_3 + 4z_1z_3 + 9z_1z_2}| = |z_1z_2z_3|}$

$\mathbf{|z_2z_3 + 4z_1z_3 + 9z_1z_2| = |z_1|\,|z_2||z_3|}$

$\mathbf{\Rightarrow |z_2z_3 + 4z_1z_3 + 9z_1z_2| = 6}$

Exercise 2.5 (9): **S.T** $\mathbf{z^3 + 2\bar{z} = 0}$ **has five solution .**

Solution :

$z^3 + 2\bar{z} = 0$ $\qquad$ - (1)

$z^3 = -2\bar{z}$

$|z|^3 = |-2||\bar{z}| \Rightarrow |z|^3 = 2|z|$

$|z|^3 - 2|z| = 0 \Rightarrow |z|(|z|^2 - 2) = 0$

$|z| = 0$ $\quad$ & $\quad |z|^2 - 2 = 0$

$z = 0$ $\qquad |z|^2 = 2 \Rightarrow z\bar{z} = 2 \Rightarrow \bar{z} = \frac{2}{z}$

sub in (1) $\quad z^3 + 2 \cdot \frac{2}{z} = 0 \Rightarrow z^4 + 4 = 0$

$|z| = 0$ $\quad z^4 + 4 = 0$

$\Rightarrow \mathbf{z = 0}$ $\quad \mathbf{z^4 + 4 = 0}$ **gives 4 solution**

∴ It has five solution.

Exercise 2.6 (2)

$\mathbf{z = x + iy}$ **, show that locus of z , Im** $\mathbf{\left(\frac{2z+1}{iz+1}\right) = 0}$ **is**

$\mathbf{2x^2 + 2y^2 + x - 2y = 0}$

Solution : $\mathbf{z = x + iy}$

$$\frac{2z + 1}{iz + 1} = \frac{2(x + iy) + 1}{i(x + iy) + 1} = \frac{2x + i2y + 1}{ix + i^2y + 1}$$

$$= \frac{(2x+1) + i2y}{(1-y) + ix} \times \frac{(1-y) - ix}{(1-y) - ix}$$

$$= \frac{(2x+1)(1-y) - ix(2x+1) + i2y(1-y)}{(1-y)^2 + x^2}$$

$$= \left[\frac{(2x+1)(1-y) + 2xy}{(1-y)^2 + x^2}\right] + i\left[\frac{2y(1-y) - x(2x+1)}{(1-y)^2 + x^2}\right]$$

R.P $\qquad$ I.P

$$\text{Im}\left(\frac{2z+1}{iz+1}\right) = 0 \Rightarrow \frac{2y(1-y) - x(2x+1)}{(1-y)^2 + x^2} = 0$$

$2y - 2y^2 - 2x^2 - x = 0$

∴ Locus is $\mathbf{2x^2 + 2y^2 + x - 2y = 0}$

Example 2.27:

$\mathbf{z = x + iy}$ $\quad$ **arg** $\mathbf{\left(\frac{z-1}{z+1}\right) = \frac{\pi}{2}}$ **∴ Locus is** $\mathbf{x^2 + y^2 = 1}$ **.**

Solution : $\mathbf{z = x + iy}$

$$\mathbf{\frac{z-1}{z+1} = \frac{x + iy - 1}{x + iy + 1} = \frac{(x-1) + iy}{(x+1) + iy} \times \frac{(x+1) - iy}{(x+1) - iy}}$$

$$\mathbf{= \frac{(x-1)(x+1) - iy(x-1) + iy(x+1) - i^2y^2}{(x+1)^2 + y^2}}$$

$$\mathbf{= \frac{(x-1)(x+1) + y^2}{(x+1)^2 + y^2} + i\,\frac{y(x+1) - y(x-1)}{(x+1)^2 + y^2}}$$

R.P $\qquad$ I.P

$$\mathbf{arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{2} \Rightarrow tan^{-1}\left[\frac{\frac{y(x+1) - y(x-1)}{(x+1)^2 + y^2}}{\frac{(x-1)(x+1) + y^2}{(x+1)^2 + y^2}}\right] = \frac{\pi}{2}}$$

$$\mathbf{\Rightarrow \frac{y(x+1) - y(x-1)}{(x-1)(x+1) + y^2} = tan\,\frac{\pi}{2} = \infty}$$

$\mathbf{\Rightarrow}$ **Dr** $\mathbf{= 0}$ $\qquad$ **i.e** $\mathbf{(x-1)(x+1) + y^2 = 0}$

$\mathbf{x^2 - 1 + y^2 = 0}$

$\mathbf{x^2 + y^2 = 1}$

Exercise 2.7. (6)

If $z = x + iy$ arg $\left(\frac{z-i}{z+2}\right) = \frac{\pi}{4}$ S.T. $x^2 + y^2 + 3x - 3y + 2 = 0$.

Solution : $z = x + iy$

$$\frac{z-i}{z+2} = \frac{x+iy-i}{x+iy+2} = \frac{x+i(y-1)}{(x+2)+iy} = \frac{x+i(y-1)}{(x+2)+iy} \times \frac{(x+2)-iy}{(x+2)-iy}$$

$$= \frac{x(x+2)-ixy+i(y-1)(x+2)-i^2y(y-1)}{(x+2)^2+y^2}$$

$$= \frac{x(x+2)+y(y-1)}{(x+2)^2+y^2} + i\,\frac{(y-1)(x+2)-xy}{(x+2)^2+y^2}$$

$$\arg\left(\frac{z-i}{z+2}\right) = \frac{\pi}{4} \Rightarrow \tan^{-1}\left[\frac{\frac{(y-1)(x+2)-xy}{(x+2)^2+y^2}}{\frac{x(x+2)+y(y-1)}{(x+2)^2+y^2}}\right] = \frac{\pi}{4}$$

$$\frac{(y-1)(x+2)-xy}{x(x+2)+y(y-1)} = \tan\frac{\pi}{4} = 1$$

$xy + 2y - x - 2 - xy = x^2 + 2x + y^2 - y$

$x^2 + y^2 + 2x - y - 2y + x + 2 = 0$

Locus is $x^2 + y^2 + 3x - 3y + 2 = 0$

Exercise 2.7 (4): **If $\frac{1+z}{1-z} = \cos 2\theta + i \sin 2\theta$ then $z = i\tan\theta$**

Solution : $\frac{1+z}{1-z} = \cos 2\theta + i\sin 2\theta$

$\frac{1+z}{1-z} = e^{i2\theta} \Rightarrow 1 + z = e^{i2\theta}(1 - z) = e^{i2\theta} - ze^{i2\theta}$

$z + ze^{i2\theta} = e^{i2\theta} - 1 \Rightarrow z(1 + e^{i2\theta}) = e^{i2\theta} - 1$

$z = \frac{e^{i2\theta}-1}{1+e^{i2\theta}}$ **divide nr & dr be $e^{i\theta}$**

$$z = \frac{e^{i\theta}-\frac{1}{e^{i\theta}}}{e^{i\theta}+\frac{1}{e^{i\theta}}} = \frac{e^{i\theta}-e^{-i\theta}}{e^{i\theta}+e^{-i\theta}} = \frac{\cos\theta+i\sin\theta-(\cos\theta-i\sin\theta)}{\cos\theta+i\sin\theta+\cos\theta-i\sin\theta}$$

$$z = \frac{\cos\theta+i\sin\theta-\cos\theta+i\sin\theta}{2\cos\theta} = \frac{2i\sin\theta}{2\cos\theta} \Rightarrow z = i\tan\theta$$

---

(6) $\cos\alpha + \cos\beta + \cos\gamma = \sin\alpha + \sin\beta + \sin\gamma$
S.T. $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha + \beta + \gamma)$

$\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha + \beta + \gamma)$

Solution :
$\cos\alpha + \cos\beta + \cos\gamma = 0$
$i\sin\alpha + i\sin\beta + i\sin\gamma = 0i$

$\cos\alpha + \cos\beta + \cos\gamma + i\sin\alpha + i\sin\beta + i\sin\gamma = 0 + i0$ -- (A)

let $a = \cos\alpha + i\sin\alpha = e^{i\alpha}$: $b = \cos\beta + i\sin\beta = e^{i\beta}$

$c = \cos\gamma + i\sin\gamma = e^{i\gamma}$

From (A) we get $a + b + c = 0$

$\Rightarrow a^3 + b^3 + c^3 = 3abc$

$(e^{i\alpha})^3 + (e^{i\beta})^3 + (e^{i\gamma})^3 = 3e^{i\alpha}\cdot e^{i\beta}\cdot e^{i\gamma}$

$\Rightarrow e^{i3\alpha} + e^{i3\beta} + e^{i\beta\gamma} = 3e^{i(\alpha+\beta+\gamma)}$

$\cos 3\alpha + i\sin 3\alpha + \cos 3\beta + i\sin 3\beta + \cos 3\gamma + i\sin 3\gamma$
$= 3(\cos(\alpha + \beta + \gamma) + i\sin(\alpha + \beta + \gamma))$

$(\cos 3\alpha + \cos 3\beta + \cos 3\gamma) + i(\sin 3\alpha + \sin 3\beta + \sin 3\gamma)$

$= 3[\cos(\alpha + \beta + \gamma)] + \sin(\alpha + \beta + \gamma)]$

Equating real part
$\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha + \beta + \gamma)$
$\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha + \beta + \gamma)$

Example 2.34 : solve : $z^3 + 8i = 0$

$z^3 + 8i = 0$
$z^3 = -8i$
$= 8(-i)$

$z^3 = 8\left[\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right]$

$= 8\left[\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right]$

$= 8\left[\cos\left(2k\pi - \frac{\pi}{2}\right) + i\sin\left(2k\pi - \frac{\pi}{2}\right)\right)$

$z^3 = 8\left[\cos\left(\frac{4k\pi-\pi}{2}\right) + i\sin\left(\frac{4k\pi-\pi}{2}\right)\right]$

$z = 8^{\frac{1}{3}}\left[\cos\left(\frac{4k\pi-\pi}{2}\right) + i\sin\left(\frac{4k\pi-\pi}{2}\right)\right]^{\frac{1}{3}}$

$z = (2^3)^{1/3}\left[\cos(4k - 1)\frac{\pi}{6} + i\sin(4k - 1)\frac{\pi}{6}\right]$

$k = 1, 2, 3$

$k = 0$, $z = 2\left[\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right] = 2\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)$

$= \sqrt{3} - i$

$k = 1, z = 2\left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right] = 2(0 + i) = 2i$

$k = 2,\ z = 2\left[\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right] = \left(-\frac{\sqrt{3}}{2} - \frac{i}{2}\right)$

$= -\sqrt{3} - i$

Example 2.35

Find the cube roots of $\sqrt{3} + i$

Solution :

Let $z = \sqrt[3]{\sqrt{3} + i} = (\sqrt{3} + i)^{\frac{1}{3}}$

$\sqrt{3} + i = r(\cos\theta + i\sin\theta)$

$a = \sqrt{3}$ $b = 1$ $a^2 = 3$ $b^2 = 1$
$r = \sqrt{a^2 + b^2} = \sqrt{3 + 1} = \sqrt{4} = 2$
$\alpha = \tan^{-1}\left|\frac{b}{a}\right| = \tan^{-1}\left|\frac{1}{\sqrt{3}}\right| = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$\alpha = \frac{\pi}{6}$
$\theta = \alpha = \frac{\pi}{6}$ $\sqrt{3} + i$ lies in I Quadrant

$\sqrt{3} + i = 2\left[\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right]$

$= 2\left[\cos\left(2k\pi + \frac{\pi}{6}\right) + i\sin\left(2k\pi + \frac{\pi}{6}\right)\right]$

$= 2\left[\cos\left(\frac{12k\pi+\pi}{6}\right) + i\sin\left(\frac{12k\pi+\pi}{6}\right)\right]$

$(\sqrt{3} + i)^{1/3} = 2^{1/3}\left[\cos\left(\frac{12k\pi+\pi}{6}\right) + i\sin\left(\frac{12k\pi+\pi}{6}\right)\right]^{\frac{1}{3}}$

$z = 2^{1/3}\left[\cos(12k + 1)\frac{\pi}{18} + i\sin(12k + 1)\frac{\pi}{18}\right.$

$k = 0,1,2$

$k = 0$ $z_1 = 2^{1/3}\left(\cos\frac{\pi}{18} + i\sin\frac{\pi}{18}\right) = 2^{1/3}e^{i\frac{\pi}{18}}$

$k = 1$ $z_2 = 2^{1/3}\left[\cos\frac{13\pi}{18} + i\sin\frac{13\pi}{18}\right] = 2^{1/3}e^{i\frac{13\pi}{18}}$

$k = 2$ $z_3 = 2^{1/3}\left[\cos\frac{25\pi}{18} + i\sin\frac{25\pi}{18}\right] = 2^{1/3}e^{i\frac{25\pi}{18}}$

(3) If $(x_1+iy_1)(x_2+iy_2)\ldots(x_n+iy_n)=a+ib$

show that $(x_1^2+y_1^2)(x_2^2+y_2^2)\ldots(x_n^2+y_n^2)=a^2+b^2$

$\sum_{r=1}^{n}\tan^{-1}\left(\frac{y_r}{x_r}\right)=k\pi+\tan^{-1}\left(\frac{b}{a}\right) \quad k \in z$

Solution :

(i) $(x_1+iy_1)(x_2+iy_2)\ldots(x_n+iy_n)=a+ib$

$|(x_1+iy_1)(x_2+iy_2)(x_3+iy_3)|=|a+ib|$

$|(x_1+iy_1)||(x_2+iy_2)|\ldots|(x_3+iy_3)|=|a+ib|$

$\sqrt{x_1^2+y_1^2}\sqrt{x_2^2+y_2^2}\cdots\sqrt{x_n^2+y_n^2}=\sqrt{a^2+b^2}$

On squaring : $(x_1^2+y_1^2)(x_2^2+y_2^2)\ldots(x_n^2+y_n^2)=a^2+b^2$

(ii) $\arg[(x_1+iy_1)(x_2+iy_2)\ldots(x_n+iy_n)]=\arg(a+ib)$

$\arg(x_1+iy_1)+\arg(x_2+iy_2)+\cdots+\arg(x_n+iy_n)$

$=\arg(a+ib)$

$\Rightarrow \tan^{-1}\left(\frac{y_1}{x_1}\right)+\tan^{-1}\left(\frac{y_2}{x_2}\right)+\cdots+\tan^{-1}\left(\frac{y_n}{x_n}\right)=$

$\tan^{-1}\left(\frac{b}{2}\right)$

∴ Given solution

$\tan^{-1}\left(\frac{y_1}{x_1}\right)+\tan^{-1}\left(\frac{y_2}{x_2}\right)+\cdots+\tan^{-1}\left(\frac{y_n}{x_n}\right)$

$=k\pi+\tan^{-1}\left(\frac{b}{2}\right)$

Example : 2.36.

z_1, z_2, z_3 are vertices of an equilateral traingle inscribed in the circle $|z|=2$, $z_1=1+i\sqrt{3}$.

Solution :

$|z|=2$ represents circle with center $(0,0)$ and radius $=2$

z_1, z_2, z_3 lies on circle and forms a vertices of equilateral triangle.

z_2, z_3 obtained by rotating $z_1=1+i\sqrt{3}$ by $120°, 240°$ in anti clockwise direction respectively.

$z_1=1+i\sqrt{3}$

$z_2=(1+i\sqrt{3})\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$

$=(1+i\sqrt{3})\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)$

$=-\frac{1}{2}+i\frac{\sqrt{3}}{2}-\frac{i\sqrt{3}}{2}+i^2\frac{3}{2}$

$=-\frac{1}{2}-\frac{3}{2}=-\frac{4}{2}=-2$

z_3 is obtained by multiplying z_2 with $e^{i\frac{2\pi}{3}}$

$z_3=-2\left[\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right]$

$=-2\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)=\frac{2}{2}-i\,2\cdot\frac{\sqrt{3}}{2}$

$=1-i\sqrt{3}$

∴ $z_2=-2$; $z_3=1-i\sqrt{3}$

$x+\frac{1}{x}=2\cos\alpha \Rightarrow x^2+1=2\cos\alpha x$

$x^2-2\cos\alpha x+\cos^2\alpha+\sin^2\alpha=0$

$(x-\cos\alpha)^2=-\sin^2\alpha \Rightarrow (x-\cos\alpha)^2=i^2\sin^2\alpha$

$x-\cos\alpha=\pm\sqrt{i^2\sin^2\alpha} \Rightarrow x=\cos\alpha\pm i\sin\alpha$

let $x=\cos\alpha+i\sin\alpha$

Similarly $y=\cos\beta+i\sin\beta$

4. If $2\cos\alpha=x+\frac{1}{x}$ & $2\cos\beta=y+\frac{1}{y}$

Show :

(i) $\frac{x}{y}+\frac{y}{x}=2\cos(\alpha-\beta)$

$\frac{x}{y}=\frac{\cos\alpha+i\sin\alpha}{\cos\beta+i\sin\beta}=\cos(\alpha-\beta)+i\sin(\alpha-\beta)$

$\frac{y}{x}=\left(\frac{x}{y}\right)^{-1}=[\cos(\alpha-\beta)+i\sin(\alpha-\beta)]^{-1}$

$=\cos(\alpha-\beta)-i\sin(\alpha-\beta)$

$\frac{x}{y}+\frac{y}{x}=\cos(\alpha-\beta)+i\sin(\alpha-\beta)+\cos(\alpha-\beta)-i\sin(\alpha-\beta)$

$=2\cos(\alpha-\beta)$

(ii) $xy-\frac{1}{xy}=2i\sin(\alpha+\beta)$

$xy=(\cos\alpha+i\sin\alpha)(\cos\beta+i\sin\beta)$

$=\cos(\alpha+\beta)+i\sin(\alpha+\beta)$

$\frac{1}{xy}=(xy)^{-1}=[\cos(\alpha+\beta)+i\sin(x+\beta)]^{-1}$

$=\cos(\alpha+\beta)-i\sin(\alpha+\beta)$

$xy-\frac{1}{xy}=\cos(\alpha+\beta)+i\sin(\alpha+\beta)$

$-\cos(\alpha+\beta)+i\sin(\alpha+\beta)$

$xy-\frac{1}{xy}=2i\sin(\alpha+\beta)$

(iii) $\frac{x^m}{y^n}-\frac{y^n}{x^m}=2i\sin(m\alpha-n\beta)$

$\frac{x^m}{y^n}=\frac{(\cos\alpha+i\sin\alpha)^m}{(\cos\beta+i\sin\beta)^n}=\frac{\cos m\alpha+i\sin m\alpha}{\cos n\beta+i\sin n\beta}$

$=\cos(m\alpha-n\beta)+i\sin(m\alpha-n\beta)$

$\frac{x^m}{y^n}=\left(\frac{x^m}{y^n}\right)^{-1}=[\cos(m\alpha-n\beta)+i\sin(m\alpha-n\beta)]^{-1}$

$=\cos(m\alpha-n\beta)-i\sin(m\alpha-n\beta)$

$\frac{x^m}{y^n}-\frac{y^n}{x^m}=\cos(m\alpha-n\beta)+i\sin(m\alpha-n\beta)-$

$\cos(m\alpha-n\beta)+i\sin(m\alpha-n\beta)$

$=2i\sin(m\alpha-n\beta)$

$x^my^n+\frac{1}{x^my^n}=2\cos(m\alpha+n\beta)$

$x^my^n=(\cos\alpha+i\sin\alpha)^m(\cos\beta+i\sin\beta)^n$

$=(\cos m\alpha+i\sin m\alpha)(\cos n\beta+i\sin n\beta)$

$=\cos(m\alpha+n\beta)+i\sin(m\alpha+n\beta)$

$\frac{1}{x^my^n}=(x^my^n)^{-1}$

$=[\cos(m\alpha+n\beta)+i\sin(m\alpha+n\beta)]^{-1}$

$=\cos(m\alpha+n\beta)-i\sin(m\alpha+n\beta)$

$x^my^n+\frac{1}{x^my^n}=\cos(m\alpha+n\beta)+i\sin(m\alpha+n\beta)$

$+\cos(m\alpha+n\beta)-i\sin(m\alpha+n\beta)$

$=2\cos(m\alpha+n\beta)$

2 MARKS & 3 MARKS

2 - MARKS

EXERCISE 3.1

2) (i) construct a cubic polynomial with roots 1, 2, 3

Solution : $\alpha = 1 \quad \beta = 2 \quad \gamma = 3$

$\Sigma_1 = \alpha + \beta + \gamma = 1 + 2 + 3 = 6$

$\Sigma_2 = \alpha\beta + \alpha\gamma + \beta\gamma = 1(2) + 1(3) + 2(3)$
$= 2 + 3 + 6 = 11$

$\Sigma_3 = \alpha\beta\gamma = 1(2)(3) = 6.$

$\therefore$ Eqn: $x^3 - \Sigma_1 x^2 + \Sigma_2 x - \Sigma_3 = 0$

$\therefore x^3 - 6x^2 + 11x - 6 = 0.$

2(ii) roots 1, 1, − 2

solution: $\alpha = 1 \quad \beta = 1 \quad \gamma = -2$

$\Sigma_1 = \alpha + \beta + \gamma = 1 + 1 + (-2) = 2 - 2 = 0$

$\Sigma_2 = \alpha\beta + \alpha\gamma + \beta\gamma = 1(1) + 1(-2) + 1(-2)$
$= 1 - 2 - 2 = -3$

$\Sigma_3 = \alpha\beta\gamma = 1(1)(-2) = -2.$

$\therefore$ Equation: $x^3 - \Sigma_1 x^2 + \Sigma_2 x - \Sigma_3 = 0$

$\therefore x^3 - 0x^2 + (-3)x - (-2) = 0$ I.e $x^3 - 3x + 2 = 0$

2(iii) roots 2, 1/2 and 1 .

solution: $\alpha = 2 \ \beta = \frac{1}{2}, \quad \gamma = 1.$

$\Sigma_1 = \alpha + \beta + \gamma = 2 + \frac{1}{2} + 1 = \frac{4 + 1 + 2}{2} = \frac{7}{2}$

$\Sigma_2 = \alpha\beta + \alpha\gamma + \beta\gamma = 2\left(\frac{1}{2}\right) + 2(1) + \frac{1}{2}(1)$
$= 1 + 2 + \frac{1}{2} = \frac{2 + 4 + 1}{2} = \frac{7}{2}$

$\Sigma_3 = \alpha\beta\gamma = 2\left(\frac{1}{2}\right)(1) = 1.$

$\therefore$ Equation: $x^3 - \Sigma_1 x^2 + \Sigma_2 x - \Sigma_3 = 0$

$\therefore x^3 - \left(\frac{7}{2}\right)x^2 + \frac{7}{2}(x) - 1 = 0 \Rightarrow 2x^3 - 7x^2 + 7x - 2 = 0$

---

8. If α , β , γ and δ are the roots of polynomial equation $2x^4 + 5x^3 - 7x^2 + 8 = 0$. find a quadratic equation with integer co efficients whose roots are $\alpha + \beta + \gamma + \delta$ and $\alpha\beta\gamma\delta$.

Solution: $2x^4 + 5x^3 - 7x^2 + 0x + 8 = 0$

$a = 2 \quad b = 5 \quad c = -7 \quad d = 0 \quad e = 8$

$\Sigma_1 = \alpha + \beta + \gamma + \delta = -b/a = -3/2$

$\Sigma_4 = \alpha\beta\gamma\delta = \frac{e}{a} = \frac{8}{2} = 4$

roots are $\alpha + \beta + \gamma + \delta$ and $\alpha\beta\gamma\delta$

i.e $-5/2$ and 4 .

S.O.R $= -\frac{5}{2} + 4 = \frac{-5 + 8}{2} = \frac{5}{2}$

P.O.R $= \frac{-5}{2}(4) = -10$

Eqn : $x^2 - (\text{S.O.R})x + \text{P.O.R} = 0$

$\Rightarrow x^2 - \frac{3}{2}x + (-10) = 0 \Rightarrow 2x^2 - 3x - 20 = 0$.

---

(11) A 12 metre tall tree was broken into two parts . It was found that the height of the part which was left standing was the cube root of length of the part that was cut away. Formulate this into a mathematical Problem to find the height of part which was left standing.

Solution :

let AC = 12 , AB = x , BC = 12 − x

Given: $x = \sqrt[3]{12 - x}$

$\Rightarrow x^3 = 12 - x \Rightarrow x^3 + x - 12 = 0$

EXERCISE 3.2

2) Find a polynomial equation of minimum deqree with rational co.efficients having $2 + \sqrt{3}i$ as a root

Solution : Given $2 + \sqrt{3}i$ is a root $\therefore$ other root is $2 - \sqrt{3}i$

S.O.R $= 2 + \sqrt{3}i + 2 - \sqrt{3}i = 4$

P.O.R $= (2 + \sqrt{3}i)(2 - \sqrt{3}i) = 2^2 + \sqrt{3}^2 = 4 + 3 = 7$

$\therefore$ Eqn is $x^2 - (\text{S.O.R})x + \text{P.O.R} = 0$
$\Rightarrow x^2 - 4x + 7 = 0.$

---

2) Find a polynomial equation with minimum degree with rational co.efficients having $2i + 3$ as a root

Solution : Given one root is $2i + 3 = 3 + 2i$

Other root is $3 - 2i$

Sum of the roots $= 3 + 2i + 3 - 2i = 6$

Product of roots $= (3 + 2i)(3 - 2i)$
$= 3^2 + 2^2 = 9 + 4 = 13$

Equation is $x^2 - (\text{S.O.R})x + \text{P.O.R} = 0$
$\Rightarrow x^2 - 6x + 13 = 0$

EXERCISE 3.3 (7) Solve the equation : $x^4 - 14x^2 + 45 = 0$

$(x^2)^2 - 14x^2 + 45 = 0$ let $t = x^2$

$t^2 - 14t + 45 = 0 \Rightarrow (t - 9)(t - 5) = 0$

$t - 9 = 0 \quad t - 5 = 0$

$t = 9 \quad t = 5$

$\Rightarrow x^2 = 9 \quad \Rightarrow x^2 = 5$

$\Rightarrow x = \pm\sqrt{3} \quad \Rightarrow x = \pm\sqrt{5}$

$\Rightarrow x = \pm 3$

Roots are $3 , -3 , \sqrt{5} , -\sqrt{5}$

EXERCISE 3.5: 2(ii) $x^8 - 3x + 1 = 0$

Solution :

$a_n = 1 \quad a_0 = 1$. Note : $\frac{p}{q}$ is a rational root of p(x) (p,q)=1

Then p is divisor 1 , q is divisor 1

possible values of p ± 1, Possible values of q ± 1

$\frac{p}{q}$ possible value is $\pm\frac{1}{1}$

$p(x) = x^8 - 3(x) + 1$

$p(1) = 1 - 3(1) + 1 = -1 \neq 0$

$p(-1) = (-1)^8 - 3(-1) + 1 = 1 + 3 + 1 \neq 0$

$\therefore$ no rational roots

EXERCISE 3.6

1. Discuss the maximum possible number of positive and negative roots of

$9x^9 - 4x^8 + 4x^7 - 3x^6 + 2x^5 + x^3 + 7x^2 + 7x + 2 = 0$

Solution :

$p(x) = 9x^9 - 4x^8 + 4x^7 - 3x^6 + 2x^5 + x^3 + 7x^2 + 7x + 2$

+ 1 − 2 + 3 − 4 + + + + +

No of sign changes = 4 $\quad\therefore$ max no of positive real roots = 4

$p(-x) = 9(-x)^9 - 4(-x)^8 + 4(-x)^7 - 3(-x)^6 + 2(-x)^5 + (-x)^3 + 7(-x)^2 + 7(-x) + 2$

$= -9x^9 - 4x^8 - 4x^7 - 3x^6 - 2x^5 - x^3 + 7x^2 - 7x + 2$

− − − − − − + − +

No of sign changes = 3

Max no of negative real roots = 3

Degree = 9

min. no of complex roots $= 9 - (4 + 3) = 9 - 7 = 2$

2) show that the equation $x^9 - 5x^5 + 4x^4 + 2x^2 + 1 = 0$ has atleast 6 imaginary solution.

Solution :

Let $p(x) = x^9 - 5x^5 + 4x^4 + 2x^2 + 1$

$+ \quad - \quad + \quad + \quad +$

No of sign change in $p(x) = 2$

Max no of positive real roots = 2

$p(-x) = (-x)^9 - 5(-x)^5 + 4(-x)^4 + 2(-x)^2 + 1$

$= -x^9 + 5x^5 + 4x^4 + 2x^2 + 1$

$- \quad + \quad + \quad + \quad +$ No of sign changes = 1

Max no of negative real roots = 1.

degree = 9

Max no of complex roots = 9 − (2 + 1) = 9 − 3 = 6

---

2) **Determine the number of +ive and negative roots of the equation $x^9 - 5x^8 - 14x^7 = 0$**

Solution :

Let $p(x) = x^9 - 5x^8 - 14x^7$

$+ \quad - \quad -$

Number of sign change in p(x) is 1

∴ p(x) has max 1 the real roots .

$p(-x) = (-x)^9 - 5(-x)^8 - 14(-x)^7$

$= -x^9 - 5x^8 + 14x^7$

$- \quad - \quad +$ Number of sign change in $p(-x)$ is 1

∴ p(x) has max 1 negative real roots .

Degree = 9

min number of imaginary roots = 9 − (1 + 1) = 9 − 2 = 7

---

3) Find the no of real zeros and imaginary of the polynomial $x^9 + 9x^7 + 7x^5 + 5x^3 + 3x$

Solution :

$p(x) = x^9 + 9x^7 + 7x^5 + 5x^3 + 3x$ (no sign change)

$p(-x) = -x^9 - 9x^7 - 7x^5 - 5x^3 - 3x$ (no sign change)

∴ There is no +ive & no -ive real roots

But x = 0 is a root

no of unreal or imaginary roots = 9-1=8

---

3 - MARKS

EXERCISE 3.1

3). If α , β , γ are the roots of cubic equation $x^3 + 2x^2 + 3x + 4 = 0$. Form the cubic equation whose roots are

(i) 2α , 2β , 2γ (ii) $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ (iii) −α , − β , − γ

Solution: $x^3 + 2x^2 + 3x + 4 = 0$

a = 1 b = 2 c = 3 d = 4

$\alpha + \beta + \gamma = -\frac{b}{a} = -\frac{2}{1} = -2$

$\alpha\beta + \alpha\gamma + \beta\gamma = c/a = \frac{3}{1} = 3$

$\alpha\beta\gamma = -d/a = -4/1 = -4$

(i) roots are 2α , 2β , 2γ .

$\Sigma_1 = 2\alpha + 2\beta + 2\gamma = 2(\alpha + \beta + \gamma) = 2(-2) = -4$

$\Sigma_2 = (2\alpha)(2\beta) + (2\alpha)(2\gamma) + (2\beta)(2\gamma)$

$= 4\alpha\beta + 4\alpha\gamma + 4\beta\gamma$

$= 4(\alpha\beta + \alpha\gamma + \beta\gamma) = 4(3) = 12$

$\Sigma_3 = (2\alpha)(2\beta)(2\gamma) = 8\alpha\beta\gamma = 8(-4) = -32$

Equation : $x^3 - \Sigma_1 x^2 + \Sigma_2 x - \Sigma_3 = 0$

$x^3 - (-4)x^2 + 12x - (-32) = 0$

$\Rightarrow x^3 + 4x^2 + 12x + 32 = 0$

(ii) roots are $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$

$\Sigma_1 = \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma} = \frac{3}{-4} = \frac{-3}{4}$

$\Sigma_2 = \frac{1}{\alpha}.\frac{1}{\beta} + \frac{1}{\beta}.\frac{1}{\gamma} + \frac{1}{\alpha\gamma} = \frac{\gamma + \alpha + \beta}{\alpha\beta\gamma} = \frac{2}{4}$

$\Sigma_3 = \frac{1}{\alpha} \cdot \frac{1}{\beta} \cdot \frac{1}{\gamma} = \frac{1}{\alpha\beta\gamma} = \frac{1}{-4} = \frac{-1}{4}$

Equation : $x^3 - \Sigma_1 x^2 + \Sigma_2 x - \Sigma_3 = 0$

$x^3 - \left(\frac{-3}{4}\right)x^2 + \frac{2}{4}x - \left(-\frac{1}{4}\right) = 0 \quad \Rightarrow 4x^3 + 3x^2 + 2x + 1 = 0.$

(iii) roots are −α , − β , − r

$\Sigma_1 = (-\alpha) + (-\beta) + (-\gamma) = -(\alpha + \beta + \gamma) = -(-2) = 2$

$\Sigma_2 = (-\alpha)(-\beta) + (-\alpha)(-\gamma) + (-\beta)(-\gamma)$

$= \alpha\beta + \alpha\gamma + \beta\gamma = 3$

$\Sigma_3 = (-\alpha)(-\beta)(-\gamma) = -\alpha\beta\gamma = -(-4) = 4$

Equation : $x^3 - \Sigma_1 x^2 + \Sigma_2 x - \Sigma_3 = 0$

$x^3 - (+2)x^2 + 3x - 4 = 0$

$x^3 - 2x^2 + 3x - 4 = 0.$

---

5)Find the sum of the squares of the roots of the equation $2x^4 - 8x^3 + 6x^2 - 3 = 0$

Solution: $2x^4 - 8x^3 + 6x^2 + 0x - 3 = 0$

let the roots be α , β , γ , δ

a = 2 b = − 8 c = 6 d = 0 e = − 3

$\Sigma_1 = \alpha + \beta + \gamma + \delta = \frac{-b}{a} = \frac{-(-8)}{2} = 4$

$\Sigma_2 = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{c}{a} = \frac{6}{2} = 3.$

$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = (\alpha + \beta + \gamma + \delta)^2 - 2(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)$

$\Rightarrow \alpha^2 + \beta^2 + \gamma^2 + \delta^2 = (4)^2 - 2(3)$

$= 16 - 6 = 10.$

---

(7) If α, β, γ are the roots of the equation $ax^3 + bx^2 + cx + d = 0$, find $\sum \frac{\alpha}{\beta\gamma}$

Solution:

$ax^3 + bx^2 + cx + d = 0$

let the Roots be α , β , γ

$\Sigma_1 = \alpha + \beta + \gamma = -b/a$

$\Sigma_2 = \alpha\beta + \beta\gamma + \alpha\gamma = c/a$

$\Sigma_3 = \alpha\beta\gamma = -d/a$

To find: $\sum \frac{\alpha}{\beta\gamma} = \frac{\alpha}{\beta\gamma} + \frac{\beta}{\alpha\gamma} + \frac{\gamma}{\alpha\beta} = \frac{\alpha^2+\beta^2+\gamma^2}{\alpha\beta\gamma}$

$= \frac{(\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\alpha\gamma)}{\alpha\beta\gamma} = \frac{\left(-\frac{b}{a}\right)^2 - 2.\frac{c}{a}}{\frac{d}{a}} = \frac{\frac{b^2}{a^2} - \frac{2c}{a}}{\frac{d}{a}}$

$= \frac{b^2 - 2ac}{a^2} \times \frac{a}{d} = \frac{b^2 - 2ac}{ad}$

---

8. If p and q are the roots of the equation $lx^2 + nx + n = 0$ show that $\sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}} + \sqrt{\frac{n}{l}} = 0$.

Solution : $lx^2 + nx + n = 0$

a = l , b = n , c = n roots are p , q

$\Rightarrow p + q = \frac{-b}{a} = \frac{-n}{l}$

Also $pq = \frac{c}{a} = \frac{n}{l}$

Now , $\sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}} + \sqrt{\frac{n}{l}} = \frac{\sqrt{p}}{\sqrt{q}} + \frac{\sqrt{q}}{\sqrt{p}} + \sqrt{\frac{n}{l}} = \frac{p+q}{\sqrt{pq}} + \sqrt{\frac{n}{l}}$

$= \frac{-n/l}{\sqrt{\frac{n}{l}}} + \sqrt{\frac{n}{l}} = \frac{-\sqrt{\frac{n}{l}}\sqrt{\frac{n}{l}}}{\sqrt{\frac{n}{l}}} + \sqrt{\frac{n}{l}} = -\sqrt{\frac{n}{l}} + \sqrt{\frac{n}{l}} = 0$

10. If the equation $x^2+px+q=0$ & $x^2+p'x+q'=0$ have a common root Show that it is $\frac{pq'-q'p}{q-q'}$ or $\frac{q-q'}{p'-p}$.

Solution :

$x^2+px+q=0 \quad x^2+p'x+q'=0$

Let α be the common root

$\alpha^2+p\alpha+q=0$ & $\alpha^2+p'\alpha+q'=0$

$$\Rightarrow \frac{\alpha^2}{pq'-p'q}=\frac{\alpha}{q-q'}=\frac{1}{p'-p}$$

$$\Rightarrow \frac{\alpha^2}{\alpha}=\frac{pq'-p'q}{q-q'} \quad \& \quad \alpha=\frac{q-q'}{p'-p}$$

$\Rightarrow \alpha=\frac{pq'-p'q}{q-q'}$ or $\frac{q-q'}{p'-p}$.

EXERCISE 3.2:

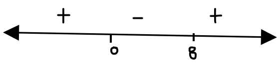
1} If k is real, discuss the nature of the roots of the polynomial equation $2x^2+kx+k=0$ interm of k.

Solution :

$2x^2+kx+k=0 \quad a=2 \quad b=k \quad c=k$

$\Delta=b^2-4ac=k^2-4(2)k=k^2-8k$

$=k(k-8)$



(i) for real and equal roots

$\Delta=0 \Rightarrow k(k-8)=0 \quad k=0 \quad k=8$

(ii) For real & distinct roots

$\Delta>0 \Rightarrow k(k-8)>0$

$\Rightarrow k\in(-\infty,0)\cup(8,\infty)$

(iii) For imaginary roots

$\Delta<0 \Rightarrow k(k-8)<0$

$\Rightarrow k\in(0,8)$

---

5) Prove that a straight line and parabola cannot intersect at more than 2 points

Solution:

Parabola eqn : $y^2=4ax$ – (1)

line eqn : $y=mx+c$ – (2)

sub (2) in (1) $(mx+c)^2=4ax$

$\Rightarrow m^2x^2+2mcx+c^2=4ax$

$\Rightarrow m^2x^2+(2mc-4a)x+c^2=0$

This is a quadratic eqn in x , x can have max 2 values .

EXERCISE : 3.3

1. Solve the cubic equation $2x^3-x^2-18x+9=0$, if sum of the two of roots vanishes.

Solution :

$2x^3-x^2-18x+9=0$

$a=2 \quad b=-1 \quad c=-18 \quad d=9$

Let the roots be α,β,γ

given $\alpha+\beta=0$

also $\alpha+\beta+\gamma=\frac{-b}{a}=\frac{-(-1)}{2}=\frac{1}{2}$

$\because \alpha+\beta=0 \Rightarrow \gamma=1/2$

$\frac{1}{2}$	2	-1	-18	9
	0	1	0	-9
	2	0	-18	0

Quad. eqn $2x^2-18=0$

$\Rightarrow x^2-9=0$

$\Rightarrow x^2=9$

$\Rightarrow x=\pm 3$

roots are $3,-3,\frac{1}{2}$.

---

2. Solve the equation $9x^3-36x^2+44x-16=0$ if the roots form an arithment progression

Solution : $9x^3-36x^2+44x-16=0$

$a=9 \quad b=-36 \quad c=44 \quad d=-16$

Let the roots be α,β,γ be in . A.P

$\alpha=a_1-d,\ \beta=a_1,\ r=a_1+d$

$$S.O.R=a_1-d+a_1+a_1+d=\frac{-b}{a}=\frac{-(-36)}{9}$$

$\Rightarrow 3a_1=4 \quad \Rightarrow a_1=4/3$

$\frac{4}{3}$	9	-36	44	-16
	0	12	-32	16
	9	-24	12	0

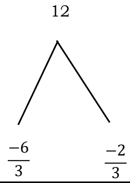
Quadratic equation is

$9x^2-24x+12=0$

$\div 3 \quad 3x^2-8x+4=0$

$(x-2)\left(x-\frac{2}{3}\right)=0 \quad \therefore x=2, 2/3$

$\therefore$ Roots are $2,\frac{2}{3},\frac{4}{3}$



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2. Solve the equation $3x^3-26x^2+52x-24=0$ if the roots form a geometric progression .

Solution : $3x^3-26x^2+52x-24=0$

$a=3 \quad b=-26 \quad c=52 \quad d=-24$

let the roots be α,β,γ .root are in G.P. $\alpha=\frac{a_1}{r} \quad \beta=a_1 \quad \gamma=a_1r$

Product of roots $=\frac{a_1}{r}\cdot a_1\cdot a_1r=\frac{-(d)}{a}$

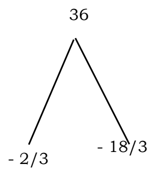
$$\Rightarrow a_1^3=\frac{-(-24)}{3}=\frac{24}{3} \Rightarrow a_1^3=8 \Rightarrow a_1=2$$

2	3	-26	52	-24
	0	6	-40	24
	3	-20	12	0

Quadratic eqn $3x^2-20x+12=0$

$\Rightarrow \left(x-\frac{2}{3}\right)(x-6)=0$

$\Rightarrow x=\frac{2}{3},6 \quad \therefore$ roots are $\frac{2}{3},2,6$



6.Solve the equation

Solution :

(i) $2x^3-9x^2+10x=3$, $\quad \mathbf{2+(-9)+10+(-3)=12-12=0}$

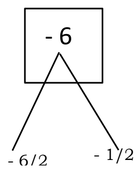
$2x^3-9x^2+10x-3=0$ ($\therefore$ x =1 is a root)

1	2	-9	10	-3
	0	2	-7	3
	2	-7	3	0

Quadratic eqn $2x^2-7x+3=0$

$(x-3)\left(x-\frac{1}{2}\right)=0 \Rightarrow x=3, x=\frac{1}{2}$

Roots are $1,3,\frac{1}{2}$



(ii) $8x^3-2x^2-7x+3=0 \quad \mathbf{8+(-2)+(-7)+3=11-9=2\neq 0}$

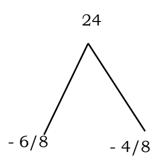
But $8+(-7)=1 \quad -2+3=1$ (x = -1 is a root)

-1	8	-2	-7	3
	0	-8	10	-3
	8	-10	3	0

Quadratic equation .

$$8x^2-10x+3=0 \Rightarrow \left(x-\frac{3}{4}\right)\left(x-\frac{1}{2}\right)=0$$

$x=\frac{3}{4} \quad x=\frac{1}{2} \quad$ Roots are $-1,\frac{3}{4},\frac{1}{2}$



Exercise 3.5

1 (i) Solve : $\sin^2 x - 5\sin x + 4 = 0$.

Solution :

$\sin^2 x - 5\sin x + 4 = 0$

Let $t = \sin x$

$\therefore\ t^2 - 5t + 4 = 0$

$(t-4)(t-1) = 0$

$\Rightarrow t - 4 = 0 \qquad t - 1 = 0$

$\Rightarrow t = 4 \qquad t = 1$

$\Rightarrow \sin x = 4 \qquad \sin x = 1$

Not possible $\quad \sin x = \sin\frac{\pi}{2}$

$\Rightarrow x = n\pi + (-1)^n \frac{\pi}{2}$

$n \in x$.

---

(1) (ii): Solve: $12x^3 + 8x = 29x^2 - 4$

Solution :

$12x^3 + 8x = 29x^2 - 4 \Rightarrow 12x^3 - 29x^2 + 8x + 4 = 0$

1 and -1 are not roots of above equation

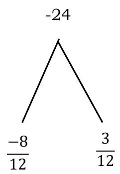
2	12	-29	8	4
	0	24	-10	-4
	12	-5	-2	0

Quad. eqn $12x^2 - 5x - 2 = 0$

$\left(x - \frac{2}{3}\right)\left(x + \frac{1}{4}\right) = 0$

$x - \frac{2}{3} = 0 \qquad x + \frac{1}{4} = 0$

$x = \frac{2}{3}, -\frac{1}{4}$



roots are 2 , $\frac{2}{3}, -\frac{1}{4}$.

---

2 (i).Examine for the rational roots of $2x^3 - x^2 - 1 = 0$

Solution :

$a_n = 2 \quad a_0 = -1$ Rational root theorem ,

$\frac{p}{q}$ is a root of polynomial $(p, q) = 1$

p must divide a_0 , q must divide a_n

$a_0 = -1$ divisor of a_0 is $-1, 1$.

$a_n = 2$ divisor of a_n is $1, -1, 2, -2$

$\frac{p}{q}$ possible values $\pm\frac{1}{1}$, $\pm\frac{1}{2}$

1	2	-1	0	-1
	0	2	1	1
	2	1	1	0

$2x^2 + x + 1 = 0 \qquad X = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\Rightarrow x = \dfrac{-1 \pm \sqrt{1 - 4(1)(2)}}{2} = \dfrac{-1 \pm \sqrt{1-8}}{2}$ $\frac{1}{8}$ $\frac{1}{8}$

$= \dfrac{-1 \pm \sqrt{-7}}{2} = \dfrac{-1 \pm \sqrt{7}i}{2}$ /

x = 1 is the only rational root

---

(3): Solve: $8x^{3/2n} - 8x^{-3/2n} = 63$

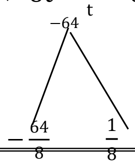
Solution :

$8x^{3/2n} - 8 \cdot x^{-3/2n} = 63$

Let $t = x^{3/2n} \Rightarrow 8t - 8 \cdot t^{-1} = 63 \Rightarrow 8t - \frac{8}{t} = 63$

$\Rightarrow 8t^2 - 8 = 63t$

$\Rightarrow 8t^2 - 63t - 8 = 0$



$\Rightarrow (t-8)\left(t + \frac{1}{8}\right) = 0$

$\Rightarrow t - 8 = 0 \qquad t + \frac{1}{8} = 0$

$\Rightarrow t = 8 \qquad t = -\frac{1}{8}$

$\Rightarrow x^{3/2n} = 2^3 \qquad x^{3/2n} = \left(\frac{-1}{2}\right)^3$

$\Rightarrow x = (2^3)^{\frac{2n}{3}} \qquad x = \left[\left(\frac{-1}{2}\right)^3\right]^{\frac{2n}{3}}$

$= 2^{2n} = 4^n \qquad x = \left(\frac{-1}{2}\right)^{2n} = \frac{1}{4^n}$

---

(4): Solve : $2\sqrt{\frac{x}{a}} + 3\sqrt{\frac{a}{x}} = \frac{b}{a} + \frac{ba}{b}$.

Solution : $\quad 2\sqrt{\frac{x}{a}} + 3\sqrt{\frac{a}{x}} = \frac{b}{a} + \frac{ba}{b}$

$2t + 3 \cdot \frac{1}{t} = \frac{b}{a} + \frac{ba}{b} \qquad t = \sqrt{\frac{x}{a}}$

$\Rightarrow 2t^2 + 3 = \left(\frac{b}{a} + \frac{6a}{b}\right) \cdot t \qquad \frac{1}{t} = \sqrt{\frac{a}{x}}$

$\Rightarrow 2t^2 - \left(\frac{b}{a} + \frac{6a}{b}\right)t + 3 = 0$

$\Rightarrow 2t^2 - \frac{b}{a}t - \frac{6a}{b}t + 3 = 0 \Rightarrow t\left(2t - \frac{b}{a}\right) - \frac{3a}{b}\left(2t - \frac{b}{a}\right) = 0$

$\Rightarrow \left(t - \frac{3a}{b}\right)\left(2t - \frac{b}{a}\right) = 0 \Rightarrow t - \frac{3a}{b} = 0 \qquad 2t - \frac{b}{a} = 0$

$\Rightarrow t = \frac{3a}{b} \qquad 2t = \frac{b}{a}$

$\Rightarrow \sqrt{\frac{x}{a}} = \frac{3a}{b} \qquad \sqrt{\frac{x}{a}} = \frac{b}{2a}$

$\Rightarrow \frac{x}{a} = \frac{9a^2}{b^2} \qquad \Rightarrow \frac{x}{a} = \frac{b^2}{4a^2}$

$\Rightarrow x = \frac{9a^3}{b^2} \qquad \Rightarrow x = \frac{b^2}{4a} \qquad \therefore$ soln : $\frac{9a^3}{b^2}, \frac{b^2}{4a}$

5(ii) Solve: $x^4 + 3x^3 - 3x - 1 = 0$

Solution : $\quad x^4 + 3x^3 - 3x - 1 = 0$

1	1	3	0	-3	-1
	0	1	4	4	1
-1	1	4	4	1	0
	0	-1	-3	-1	
	1	3	1	0	

Quadralic eqn: $\quad x^2 + 3x + 1 = 0 \quad \left[\ X = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\ \right]$

$x = \frac{-3 \pm \sqrt{3^2 - 4(1)(1)}}{2} = \frac{-3 \pm \sqrt{9-4}}{2} = \frac{-3 \pm \sqrt{5}}{2}$

roots are 1 , -1 , $\frac{-3+\sqrt{5}}{2}$, $\frac{-3-\sqrt{5}}{2}$

---

(6): Find all the real numbers satisfying

$4^x - 3(2^{x+2}) + 2^5 = 0$

Solution : $\quad 4^x - 3(2^{x+2}) + 2^5 = 0$

$(2^2)^x - 3 \cdot 2^x \cdot 2^2 + 32 = 0$

$(2^x)^2 - 3(4) \cdot 2^x + 32 = 0$

Let $2^x = t \Rightarrow t^2 - 12t + 32 = 0$

$\Rightarrow (t-8)(t-4) = 0$

$\Rightarrow t - 8 = 0 \qquad t - 4 = 0$

$\Rightarrow t = 8 \qquad t = 4$

$\Rightarrow 2^x = 2^3 \qquad 2^x = 2^2$

$\Rightarrow x = 3 \qquad x = 2$

CHAPTER 5 - 2 DIMENSIONAL ANALYTICAL GEOMETRY (ONLY 5 MARKS)

Exercsie 5.1 (6).

Find the equation of the circle through the points $(1,0), (-1,0)$, and $(0,1)$.

Solution:

Let the required circle be

$x^2 + y^2 + 2gx + 2fy + c = 0$ ---(A)

The circle passes through (1,0), $(-1,0)$ and (0,1)

$(1,0) \Rightarrow 1 + 0 + 2g(1) + 2f(0) + c = 0$

$2g + c = -1$…………(1)

$(-1,0) \Rightarrow 1 + 0 + 2g(-1) + 2f(0) + c = 0$

$-2g + c = -1$……. (2)

$(0,1) \Rightarrow 0 + 1 + 2g(0) + 2f(1) + c = 0$

$2f + c = -1$………. (3)

Now solving (1), (2) and (3) .

$2g + c = -1$ ----(1)

$-2g + c = -1$ -----(2)

(1) +(2) $\Rightarrow 2c = -2 \Rightarrow c = -1$

Substituting $c = -1$ in (1) we get

$2g - 1 = -1$

$2g = -1 + 1 = 0 \Rightarrow g = 0$

Substituting $c = -1$ in (3) we get

$2f - 1 = -1 \Rightarrow 2f = -1 + 1 = 0 \Rightarrow f = 0$

So we get $g = 0, f = 0$ and $c = -1$

So the required circle will be

$x^2 + y^2 + 2(0)x + 2(0)y - 1 = 0$

(i.e) $x^2 + y^2 - 1 = 0 \Rightarrow x^2 + y^2 = 1$

Example 5.10

Find the equation of the circle passing through the points $(1,1), (2,-1)$, and $(3,2)$.

Solution

Let the general equation of the circle be

$x^2 + y^2 + 2gx + 2fy + c = 0.$ ----(1)

It passes through points (1,1), $(2,-1)$ and (3,2).

Therefore, $2g + 2f + c = -2$ ----(2)

$4g - 2f + c = -5$ -----(3)

$6g + 4f + c = -13$ ----(4)

(2)-(3) gives $-2g + 4f = 3$ ---(5)

(4)-(3) gives $2g + 6f = -8$ ---(6)

(5) + (6) gives $f = -\frac{1}{2}$

Substituting $f = -\frac{1}{2}$ in (6),$g = -\frac{5}{2}$

Substituting $f = -\frac{1}{2}$ and $g = -\frac{5}{2}$ in (2),c=4

Therefore, the required equation of the circle is

$x^2 + y^2 + 2\left(-\frac{5}{2}\right)x + 2\left(-\frac{1}{2}\right)y + 4 = 0$

$\Rightarrow x^2 + y^2 - 5x - y + 4 = 0$

Example 5.17:

Find the vertex , focus , directrix, and length of Latus rectum of $x^2 - 4x - 5y - 1 = 0$

Solution :

$x^2 - 4x - 5y - 1 = 0$

$x^2 - 4x = 5y + 1$

$x^2 - 4x + 4 = 5y + 4 + 1$

$(x-2)^2 = 5y + 5$

$(x-2)^2 = 5(y+1)$

$X = x - 2 \quad Y = y + 1 \quad 4a = 5 \Rightarrow a = \frac{5}{4}$

$X^2 = 5y$

Parabola open upward .

vertex = $(2,-1)$=(h,k) { x-2 = 0; y+1 = 0}

Focus : (0,a)⇒[(h,k+a)] $= \left(2, -1 + \frac{5}{4}\right) = \left(2, \frac{1}{4}\right)$

Eqn of directrix: Y= -a [y = k -a]

$y = -1 - \frac{5}{4} = -\frac{9}{4}$

$y = -\frac{9}{4}$

Length of Latus rectum = 4a = 5

Exercsie 5.2 - 4(iv)

Find the vertex , focus , directrix, and length of Latus rectum of $x^2 - 2x + 8y + 17 = 0$

Solution :

$x^2 - 2x + 8y + 17 = 0$

$x^2 - 2x = -8y - 17$

$x^2 - 2x + 1 = -8y - 17 + 1$

$(x-1)^2 = -8y - 16$

$(x-1)^2 = -8(y+2)$

$X = x - 1 \quad Y = y + 2 \quad 4a = 8 \Rightarrow a = 2$

$X^2 = -8Y$ Parabola open downward

Vertex(0,0) = (1,-2)=(h,k) $\{x - 1 = 0, \; y + 2 = 0\}$

Focus (0, -a)
$(h + 0, k - a) = (1, -4)$

Equation of Latusrectum (Y = -a) :

$y + 2 = -2 \Rightarrow \quad y = -4$

Equation of directrix Y = a : $y + 2 = 2 \Rightarrow y = 0$

Length of latus rectum 4a = 8

Ex 5.2 4(v)

Find the vertex, focus, directrix and length of Latus rectum of $y^2 - 4y - 8x + 12 = 0$

Solution :

$y^2 - 4y - 8x + 12 = 0$

$y^2 - 4y = -8x - 12$

$y^2 - 4y + 4 = -8x - 12 + 4$

$(y-2)^2 = -8x - 8$

$(y-2)^2 = -8(x+1)$

$Y^2 = -8X$

$X = x - 1 \quad Y = y - 2 \quad 4a = 8 \Rightarrow a = 2$

Parabola open left ward

Vertex(0,0) = (-1,2)=(h,k) $\{x + 1 = 0, x = -1;$
$y - 2 = 0, y = 2\}$

Focus (-a, 0)
$(h - a, k + 0) = (-3,2)$ {h − a = -1-2, k + 0 = 2+0 }

Eqn of directrix : X = a

$x + 1 = 2 \quad x = 2 - 1 = 1 \quad x = 1$

Length of latus rectum 4a = 8

EXAMPLE 5.20

Find the vertex, focus, length of major and minor axis of $4x^2+36y^2+40x-288y+532=0$

Solution :

$4x^2+36y^2+40x-288y+532=0$

$4x^2+40x+36y^2-288y=-532$

$4(x^2+10x)+36(y^2-8y)=-532$

$4(x^2+10x+25)+36(y^2-8y+16)=-532+100+576$

$4(x+5)^2+36(y-4)^2=144$

$\div 144 \quad \frac{4(x+5)^2}{144}+\frac{36(y-4)^2}{144}=1$

$\frac{(x+5)^2}{36}+\frac{(y-4)^2}{4}=1$ Major axis X-axis:

$X=x+5 \qquad Y=y-4$

$\frac{x^2}{36}+\frac{y^2}{4}=1$, $\{a^2=36 \Rightarrow a=6 \quad b^2=4 \Rightarrow b=2\}$

$c^2=a^2-b^2=36-4=32$, $c=\sqrt{32}$

center $(-5,4)$

Foci: $(h\pm c,k)=(-5\pm4\sqrt{2},4)$

i.e. $(-5+4\sqrt{2},4)$; $(-5-4\sqrt{2},4)=$

vertices$(h\pm a,k)$: $(-5\pm6,4)$ i.e $(1,4)$; $(-11,4)$

length of major axis $=2a=2(6)=12$

length of minor axis $=2b=2(2)=4$.

EXAMPLE 5.21

For the ellipse $4x^2+y^2+24x-2y+21=0$

Find center, vertices, foci. Also prove $L\cdot L\cdot R=2$

Solution :

$4x^2+y^2+24x-2y+21=0$; $4x^2+24x+y^2-2y=-21$

$4(x^2+6x)+1(y^2-2y)=-21$

$4(x^2+6x+9)+1(y^2-2y+1)=-21+36+1=16$

$4(x+3)^2+(y-1)^2=16 \Rightarrow \div 16 \quad \frac{4(x+3)^2}{16}+\frac{(y-1)^2}{16}=1$

$\frac{(x+3)^2}{4}+\frac{(y-1)^2}{16}=1 \quad X=x+3 \quad Y=y-1$

$\frac{X^2}{4}+\frac{Y^2}{16}=1$ Major axis Y-axis

$a^2=16 \quad a=4 \quad b^2=4 \quad b=2$

$c^2=a^2-b^2=16-4=12 \Rightarrow c=\sqrt{12}=2\sqrt{3}$

Center $(-3,1)$

vertices $(h\pm a,k)$: $(-3,1\pm4)=(-3,3)$; $(-3,-3)$

Foci $(h\pm c,k)$: $(-3,1\pm2\sqrt{3})=(-3,1+2\sqrt{3})$; $(-3,1-2\sqrt{3})$

Length of major axis $2a=8$

Length of minor axis $2b=2(2)=4$

Length of latus rectum $=2\frac{b^2}{a}=2\frac{4}{4}=2$

Ex 5.2 - 8(v)

Identify type of conic and find center , foci , vertices and directrices of $18x^2+12y^2-144x+48y+120=0$

Solution :

$18x^2+12y^2-144x+48y+120=0$

$18x^2+12y^2-144x+48y=-120$

$18(x^2-8x)+12(y^2+4y)=-120$

$18(x^2-8x+16)+12(y^2+4y+4)$

$=-120+288+48=216$

$\div 216 \quad \frac{18(x-4)^2}{216}+\frac{12(y+2)^2}{216}=1$

$\frac{(x-4)^2}{12}+\frac{(y+2)^2}{18}=1 \quad X=x-4 \quad Y=y+2$

$\frac{X^2}{12}+\frac{Y^2}{18}=1$ Major axis parallel to y-axis ;

$(a^2=18$, $a=\sqrt{18}=3\sqrt{2}$ & $b^2=12 \quad b=\sqrt{12}=2\sqrt{3})$

$c=\sqrt{a^2-b^2}=\sqrt{18-12}=\sqrt{6}$

$\frac{a}{e}=\frac{a^2}{c}=\frac{18}{\sqrt{6}}=\frac{3.\sqrt{6}.\sqrt{6}}{\sqrt{6}}=3\sqrt{6}$

Center $(4,-2)$

Vertices $(h, k\pm a)=(4,-2\pm3\sqrt{2})=(4,-2+3\sqrt{2});(4,-2-3\sqrt{2})$

Foci$(h, k\pm c)=(4,-2\pm\sqrt{6})=(4,-2+\sqrt{6});(4,-2-\sqrt{6})$

Eqn of directrices: $Y=\pm\frac{a}{e} \Rightarrow y+2=\pm3\sqrt{6}$

i.e $y=-2+3\sqrt{6}$, $y=-2-3\sqrt{6}$

Find, eccentricity, center, vertices , foci of $36x^2+4y^2-72x+32y-44=0$

Solution :

$36x^2+4y^2-72x+32y-44=0$

$36x^2-72x+4y^2+32y=44$

$36(x^2-2x)+4(y^2+8y)=44$

$36(x^2-2x+1)+4(y^2+8y+16)=44+36+64=144$

$\div 225 \quad \frac{36(x-1)^2}{144}+4\frac{(y+4)^2}{144}=1$

$\frac{(x-1)^2}{4}+\frac{(y+4)^2}{36}=1 \quad X=x-1 \quad Y=y+4$

$\frac{X^2}{4}+\frac{Y^2}{36}=1$ Major axis parallel to Y-axis

$\{a^2=36 \Rightarrow a=6, \; b^2=4 \Rightarrow b=2\}$

$c^2=a^2-b^2=36-4=32$

$\Rightarrow c=\pm\sqrt{32}=\pm\sqrt{4x4x2}=\pm4\sqrt{2}$

center $=(1,-4)$

vertices $(h, k\pm a)=(1,-4\pm6)=(1,-4+6),(1,-4-6)$

$=(1,2),(1,-10)$

Foci $(h, k\pm c)=(1,-4\pm\sqrt{32})$

$=(1,-4+4\sqrt{2});(1,-4-4\sqrt{2})$

$e=\frac{c}{a}=\frac{4\sqrt{2}}{6}=\frac{2\sqrt{2}}{3}$

Example 5.24

Find the centre, foci and e of hyperbola
$\mathbf{11x^2 - 25y^2 - 44x + 50y - 256 = 0}$

Solution :

$11x^2 - 25y^2 - 44x + 50y - 256 = 0$
$11x^2 - 44x - 25y^2 + 50y = 256$
$11(x^2 - 4x) - 25(y^2 - 2y) = 256$
$11(x^2 - 4x + 4) - 25(y^2 - 2y + 1) = 256 + 44 - 25$
$11(x-2)^2 - 25(y-1)^2 = 275$

$\div 275 \quad \frac{11(x-2)^2}{275} - \frac{25(y-1)^2}{275} = 1$

$\Rightarrow \frac{(x-2)^2}{25} - \frac{(y-1)^2}{11} = 1 \quad X = x - 2 \quad Y = y - 1$

$\frac{x^2}{25} - \frac{y^2}{11} = 1$ Transverse axis parallel to x-axis

$a^2 = 25 \Rightarrow a = 5$ & $b^2 = 11 \quad b = \sqrt{11}$

$c^2 = a^2 + b^2 = 25 + 11 = 36 \Rightarrow c = \pm 6$

centre $= (2,1)$

Foci $(h \pm ae, k) = (2 \pm 6,1) = (2+6,1); (2-6,1)$

$= (8,1); (-4,1)$

$e = \frac{c}{a} = \frac{6}{5}$

Ex 5.4 (3)

Show that the line $x - y + 4 = 0$ touches Ellipse $x^2 + 3y^2 = 12$. Also find the co. ordinates of point of contact.

Solution:

$x - y + 4 = 0 \qquad x^2 + 3y^2 = 12$

$-y = -x - 4 \qquad \frac{x^2}{12} + \frac{3y^2}{12} = 1$

$\Rightarrow y = x + 4 \qquad \frac{x^2}{12} + \frac{y^2}{4} = 1$

$m = 1 \quad c = 4$

$a^2 = 12 \quad b^2 = 4$

condition: $c^2 = a^2m^2 + b^2$

L.H.S $\quad c^2 = 4^2 = 16$

R.H.S: $\quad a^2m^2 + b^2 = 12(1) + 4 = 12 + 4 = 16$

L.H.S=R.H.S

$\therefore$ line touches Ellipse

Point of contact $= \left(-\frac{a^2m}{c}, \frac{b^2}{c}\right)$

$= \left(-\frac{12(1)}{4}, \frac{4}{4}\right) = (-3,1)$

Exercise 5.2 - 8(vi)

Identify the conic and find centre, foci, vertices and directrices of $9x^2 - y^2 - 36x - 6y + 18 = 0$

solution :

$9x^2 - y^2 - 36x - 6y + 18 = 0$

$9x^2 - 36x - y^2 - 6y = -18$

$9(x^2 - 4x) - (y^2 + 6y) = -18$

$9(x^2 - 4x + 4) - (y^2 + 6y + 9) = -18 + 36 - 9$
$9(x-2)^2 - (y+3)^2 = 9$

$\div 9 \quad \frac{(x-2)^2}{1} - \frac{(y+3)^2}{9} = 1 \quad X = x - 2 \quad Y = y + 3$

$\frac{X^2}{1} - \frac{Y^2}{9} = 1$ Transverse axis parallel to x - axis

$\{a^2 = 1 \Rightarrow a = 1;\ b^2 = 9 \Rightarrow b = 3\}$

$c^2 = a^2 + b^2 = 1 + 9 = 10 \Rightarrow c = \sqrt{10}$

centre $= (2, -3)$

Vertices $(h \pm a, k) = (2 \pm 1, -3) = (2+1, -3); (2-1, -3)$

$= (3, -3); (1, -3)$

Foci $(h \pm a, k) = (2 \pm \sqrt{10}, -3) = (2+\sqrt{10}, -3); (2-\sqrt{10}, -3)$

Eqn of directrices $X = \pm \frac{a}{e}$: $\quad \{\frac{a}{e} = \frac{a^2}{c} = \frac{1}{\sqrt{10}}\}$

$x - 2 = \pm \frac{1}{\sqrt{10}} \Rightarrow x = 2 + \frac{1}{\sqrt{10}}$, $x = 2 - \frac{1}{\sqrt{10}}$

CREATED.

Prove that the line $5x + 12y = 9$ touches $x^2 - 9y^2 = 9$. Find point of contact.

Solution:

$5x + 12y = 9 \qquad x^2 - 9y^2 = 9$

$\Rightarrow 12y = -5x + 9 \qquad \frac{x^2}{9} - \frac{y^2}{1} = 1$

$\Rightarrow y = \frac{-5}{12}x + \frac{9}{12} \qquad a^2 = 9 \quad b^2 = 1$

$\Rightarrow y = \frac{-5}{12}x + \frac{3}{4} \qquad m = -\frac{5}{12} \quad c = \frac{3}{4}$

Condition: $c^2 = a^2m^2 - b^2$

L.H.S: $c^2 = \left(\frac{3}{4}\right)^2 = \frac{9}{16}$

R.H.S: $a^2m^2 - b^2 = 9\left(\frac{-5}{12}\right)^2 - 1 = 9\left(\frac{25}{144}\right) - 1$

$= \frac{225-144}{144} = \frac{81}{144} = \frac{9}{16}$

LHS=RHS Line touch hyperbola

pt of contact $= \left(-\frac{a^2m}{c}, -\frac{b^2}{c}\right)$

$= \left(\frac{-9\left(-\frac{5}{12}\right)}{\frac{3}{4}}, \frac{-1}{\frac{3}{4}}\right) = \left(5, -\frac{4}{3}\right)$

Exercise 5.5.(1)

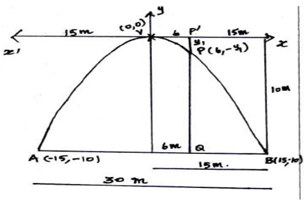
A bridge has a parabolic arch that is $10\ \text{m}$ high in the centre and $30\ \text{m}$ wide at the bottom. Find the height of the arch $6\ \text{m}$ from the centre, on either sides.

solution:

Vertex (0,0)   Parabola open downward

Equation $x^2 = -4ay$

Pt $B(15, -10)$ lies on parabola



$\therefore\ (15)^2 = -4a(-10)$

$\Rightarrow\ 225 = 4a(10)\ \Rightarrow\ 4a = \frac{225}{10}$

$\therefore$ EQN $\quad x^2 = -\frac{225}{10}y$

$\Rightarrow\ x^2 = -\frac{45}{2}y$

PQ is the height of arch 6 m to the right from center.

$PP' = y_1$

$\therefore P(6, -y_1)$ lies on parabola : $x^2 = -\frac{45}{2}y$

$$\Rightarrow\ 6^2 = -\frac{45}{2}(-y_1)$$

$$\Rightarrow\ y_1 = \frac{2 \times 36}{45} = \frac{8}{5}$$

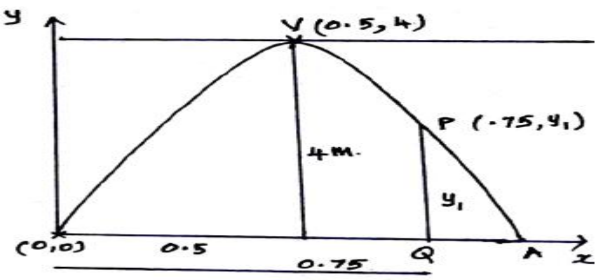
$$y_1 = 1.6\ \text{m}$$

$\therefore$ Height of arch $= 10 - y_1 = 10 - 1.6 = 8.4\ \text{m}$

Exercise 5.5.(3)

At a water fountain, water attains a maximum height of $4\ \text{m}$ at horizontal distance of $0.5\ \text{m}$ from its origin. If the path of water is a parabola, find the height of water at a horizontal distance of $0.75\ \text{m}$ from the point of origin.

SOLUTION:



From the diagram $V(0.5,4) = (h,k)$

Parabola open downward.

Equation: $(x-h)^2 = -4a(y-k)\ \Rightarrow (x-0.5)^2 = -4a(y-4$

(0,0) lies on parabola

$(0-0.5)^2 = -4a(0-4) \Rightarrow (-0.5)^2 = 4a(4) \Rightarrow 4a = \frac{0.25}{4}$

$\therefore$ Eqn: $(x-0.5)^2 = -\frac{0.25}{4}(y-4)$

Let $OQ = 0.75$

$PQ = y_1$;

$\therefore P(0.75, y_1)$ lies on parabola.

$(x-0.5)^2 = -\dfrac{0.25}{4}(y-4)$

$(0.75-0.5)^2 = \frac{-0.25}{4}(y_1 - 4)$

$\Rightarrow\ (0.25)^2 = -\frac{0.25}{4}(y_1 - 4)$

$y_1 - 4 = \frac{-4 \times (0.25)^2}{0.25}$

$y_1 - 4 = -4 \times 0.25 = -1$

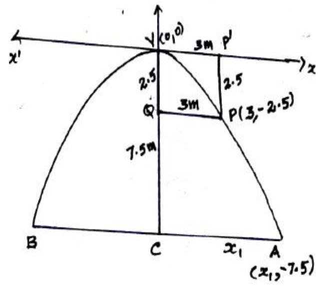
Height of water $= y_1 = -1 + 4 = 3\text{m}$.

Exercise 5.5.(8)

4) Assume that water issuing from the end of a horizontal pipe, $7.5\ \text{m}$ above the ground, describes a parabolic path. The vertex of the parabolic path is at the end of the pipe . At a position 2.5m below the outward $3\ \text{m}$ beyond the vertical line through the end of the pipe. How far beyond this vertical line will the water strike the ground?

SOLUTION:

From the diagram, vertex $V(0,0)$



Parabola open downward.

Equation $x^2 = -4ay \qquad -(1)$

Let $VP' = 3\ \text{m}$, $VQ = 2.5$

$\therefore P(3, -2.5)$ lies on parabola.

sub in (1) $3^2 = -4a(-2.5)$

$9 = 4a(2.5)$

$\Rightarrow 4a = \frac{9}{2.5}$

$\therefore$ (1) becomes $\quad x^2 = -\left(\frac{9}{2.5}\right)y$

Let $AC = x_1$ be the distance.

$\therefore A(x_1, -7.5)$ lies on parabola

$\therefore\ x_1^2 = -\left(\frac{9}{2.5}\right)(-7.5)$

$\Rightarrow x_1^2 = -9(-3)$

$x_1^2 = 27$

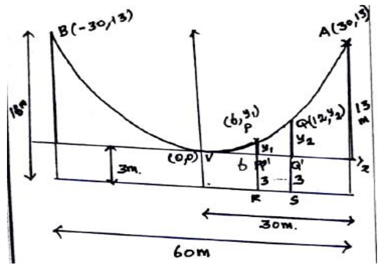
$\Rightarrow\ x_1 = \sqrt{27} = 3\sqrt{3}\ \text{m}$

Exercise 5.5 (5)

Parabolic cable of a $60\ \text{m}$ portion of the roadbed of a suspension bridge are positioned as shown below. Vertical Cables are to be spaced every 6m along this portion of the roadbed. Calculate the lengths of first two of these vertical cables from the vertex.

Solution :

From the Diagram $V(0,3)$



Parabola open upward

Equation $x^2 = 4ay$

A (30,13) lies on parabola

$30^2 = 4a(13)$

$\Rightarrow 4a = \frac{900}{13}$

$\therefore$ Eqn $x^2 = \frac{900}{13}y$

$VP' = 6,\ PP' = y_1 \quad \therefore P(6, y_1)$

$(6, y_1)$ lies on the parabola $6^2 = \frac{900}{13}y_1$

$y_1 = \frac{36 \times 13}{900} = \frac{4 \times 13}{100} = \frac{52}{100} = 0.52$

$\therefore$ Height of cable $PR = 0.52 + 3 = 3.52\text{m}$

$VQ' = 12 \qquad QQ' = y_2$

$(12, y_2)$ lies on parabola

$12^2 = \frac{900}{13}y_1$

$y_1 = \frac{144 \times 13}{900} = \frac{16 \times 13}{100} = \frac{208}{900} = 2.08$

$\therefore$ Height of cable is $= 3 + 2.08 = 5.08\text{m}$

---

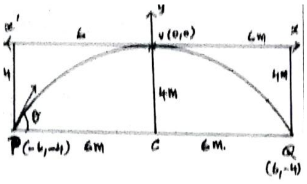
9) On lighting a rocket cracker if gets projected in a parabolic path and reaches a maximum height of 4m when it is 6m away from the point of projection Finally is reaches the ground 12 m away from the starting point . Find the angle of projection at P.

Solution :

From the diagram vertex V(0,0)

Parabola open downward

$\therefore\ x^2 = -4ay \quad -(1)$



$PC = 6\text{ m} \quad VC = 4\text{ m}$

$\therefore$ Pt $P(-6, -4)$ lies on parabola

$\therefore (-6)^2 = -4a(-4)$

$\Rightarrow\ 36 = 4a(4)$

$4a = \frac{36}{4} = 9$

$\Rightarrow$ - Becomes $x^2 = -9y$ -(2)

Let θ be the angle of projection.

To find θ, Differentiate (2) with respect to x

$2x = -9\frac{dy}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-2x}{9}$

$\Rightarrow m = \tan\theta = \frac{dy}{dx}$ at $(-6, -4) \Rightarrow \tan\theta = \frac{-2(-6)}{9} = \frac{12}{9} = \frac{4}{3}$

$\therefore$ Angle of projection at p $\theta = \tan^{-1}\left(\frac{4}{3}\right)$

4) An engineer design a satellite dish with a parabolic cross section .The dish is 5m wide at the opening , and the focus is placed 12 m from The vertex.

(i) Position a coordinate system with the origin at the vertex and the x-axis on the parabola's axis of symmetry and find an equation of the parabola .

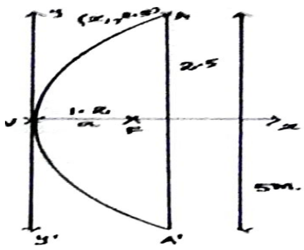
(ii) Find the depth of the satellite dish at the vertex

SOLUTION :

From the diagram

V(0, θ)

Parabola open right ward



$y^2 = 4ax$

given $a = 1.2$

$\therefore y^2 = 4(1.2)x$

$y^2 = 4.8x$

let x_1 be depth $\Rightarrow A\ (x_1, 2.5)$ lies on parabola

$(2.5)^2 = 4.8x_1 \Rightarrow$ Depth $x_1 = \frac{6.25}{4.8} = 1.3$ m.

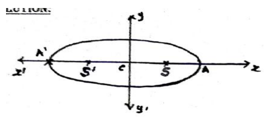
EXAMPLE 5.32: The maximum and minimum distances of the Earth from the Sun respectively are 152×10^6 km and 94.5×10^6 km . The Sun is at one focus of the elliptical orbit. Find the distance from the Sun to the other focus.

SOLUTION:

Shortest distance $= SA = 94.5 \times 10^6$

$\Rightarrow CA - CS = 94.5 \times 10^6$

$\Rightarrow a - ae = 94.5 \times 10^6$



Longest distance $= SA' = 152 \times 10^6$

$\Rightarrow CA' + CS = 152 \times 10^6$

$\Rightarrow a + ae = 152 \times 10^6$

$a + ae = 152 \times 10^6$

$a - ae = 94.5 \times 10^6$

$2ae = 57.5 \times 10^6 \quad \Rightarrow 2ae = 575 \times 10^5$Km

Distance of sun from other focus 575×10^5km

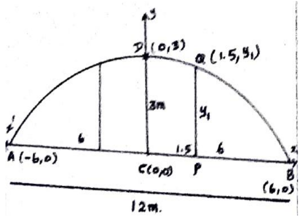
EXAMPLE 5.31: A semi elliptical archway over one way road way has and Height of 3m and width of 12m . The truch has a width of 3 m and a height of 2.7 m . Will the truck clear the opening of the archway

Solution :

Archway is in the form of Semi ellipse.

center(0,0)

Eqn : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ - (1)



Given: $AB = 2a = 12 \quad \Rightarrow a = b$

$CD = b = 3$

$\therefore$ (1) becomes

$\frac{x^2}{6^2} + \frac{y^2}{3^2} = 1 \Rightarrow \frac{x^2}{36} + \frac{y^2}{9} = 1$

Let y_1 be the height of arch 1.5m - to the right from the center.

i.e $Q\ (1.5, y_1)$ lies on ellipse

$\Rightarrow \quad \frac{(1.5)^2}{36} + \frac{y_1^2}{9} = 1 \ \Rightarrow\ \frac{y_1^2}{9} = 1 - \frac{2.25}{36}$

$\frac{y_1^2}{9} = \frac{36 - 2.25}{36}$

$\frac{y_1^2}{9} = \frac{33.75}{36} \ \Rightarrow\ y_1^2 = \frac{33.75}{36} \times 9 = \frac{33.75}{4}$

$y_1^2 = 8.43 \quad \Rightarrow\ y_1 = 2.9$m

$\because$ Heignt of truck is $2.7 < 2.9$ m

Truck will clear the opening of archway

6) Cross section of nuclear cooling tower is in the shape of a hyperbola with equation $\frac{x^2}{30^2} - \frac{y^2}{44^2} = 1$, tower is 150 m tall and the distance from the top of the tower to the centre of the hyperbola is half the distance from the base of the tower to the centre of the hyperbola . Find the diameter of the top and base of the tower

Solution :

Center (0,0) Equation of Hyperbola $\frac{x^2}{30^2} - \frac{y^2}{44^2} = 1$

Top of tower from center $= y_1$

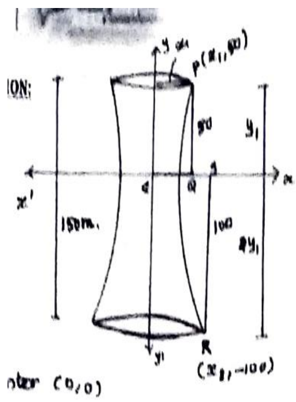
bottom of tower from center $= 2y_1$

$y_1 + 2y_1 = 150 \ \Rightarrow\ 3y_1 = 150$

$y_1 = \frac{150}{3} = 50$ m

let x_1 be the radius of top of tower

$\therefore P(x_1, 50)$ lies on Hyperbola .



$\frac{x_1^2}{30^2} - \frac{50^2}{44^2} = 1$

$\frac{x_1^2}{30^2} = 1 + \frac{2500}{1936}$

$= \frac{1936+2500}{1936} = \frac{4456}{1936}$

$x_1^2 = \frac{30^2}{44^2}(4436)$

$x_1 = \frac{30}{44}\sqrt{4436} = 45.41$

diameter $2x_1 = 2(45.41) = 90.82$

Let x_2 be the radius of bottom.

$R(x_2, -100)$ lies on hyperbola $\frac{x_2^2}{30^2} - \frac{100^2}{44^2} = 1$

$\frac{x_2^2}{30^2} = 1 + \frac{10000}{1936} = \frac{1936+10000}{1936} = \frac{11936}{1936}$

$x_2^2 = \frac{30^2}{44^2}(11936)$

$x_2 = \frac{30}{44}\sqrt{11936}$

Diameter $= 2x_2 = 148.98$m

Ex 5.5 Q.no (2)

A turnel through a mountain for o four lane highway is to have a elliptical opening . The total width of the highwoy (not the opening) is to be 16 m , and the height at the edge of the road must be sufficient for a truck 4m high to clear if the highest point of the opening is to be 5 m approximately. How wide must the opening be ?

Solution :

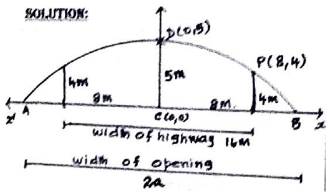
Opening of the tunnel is in elliptical shape.

Let mid pt of base be center C(0,0)

$AB = 2a =$ widh of opening

$AC = CB = a$

height $= b = 5$



Eqn of ellipse : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\frac{x^2}{a^2} + \frac{y^2}{5^2} = 1 \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{25} = 1$

width of highway $= 16$ m

At the edge, height is sufficient to clear a truck of 4 m height

$\therefore$ P(8,4) lies on ellipse

$\frac{8^2}{a^2} + \frac{4^2}{25} = 1 \Rightarrow \quad \frac{8^2}{a^2} = 1 - \frac{16}{25} = \frac{25-16}{25} = \frac{9}{25}$

$a^2 = \frac{8^2 \times 25}{9} = \frac{8^2 \times 5^2}{3^2} \Rightarrow \quad a = \frac{8 \times 5}{3} = \frac{40}{3} = 13.33 \ (\because a > 0)$

width of opening $= 2a \ = 2 \times 13.33 \quad = 26.66 \approx 26.7$ m

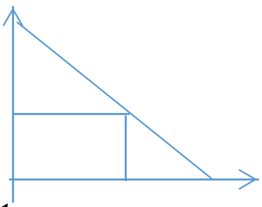
---

7) A rod of length 1.2 m moves with its ends always touching the coordinate axes. The locus of a point P on the rod, which is 0.3 m from the end in contact with x-oxis is an ellipse. Find the eccentricity.

SOLUTION:

$AB = 1.2 \quad AP = 0.3$

$BP = 1.2 - 0.3 = 0.9$



Let θ be the angle made with x-axis

Eqn : $\cos^2\theta + \sin^2\theta = 1$

$\frac{x_1^2}{(0.9)^2} + \frac{y_1^2}{(0.3)^2} = 1 \quad$ i.e $\frac{x_1^2}{0.8} + \frac{y_1^2}{0.09} = 1$

$e = \sqrt{\frac{a^2 - 1^2}{a^2}} = \sqrt{\frac{0.81 - 0.09}{0.81}} = \sqrt{\frac{0.72}{0.81}} = \sqrt{\frac{72}{81}}$

$e = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}$.

EXAMPLE 5.40: Two coast guard stations are located 600 km apart at points A(0, 0) and B(0, 600) A distres signal from a ship at P is received af slightly different times by two stations . If determined that the ship is 200 km farther from station A than it is from station B Determine the equation of hyperbola that passes through the location of the ship .

SOLUTION:

A (0,0) (0,600) Foci

Center $= \left(\frac{0+0}{2}, \frac{0+600}{2}\right) = (0,300)$

Transverse axis y-axis $\quad$ Eqn : $\frac{(y-300)^2}{a^2} - \frac{(x)^2}{b^2} = 1$

Given AB= $2ae = 600 \Rightarrow \ ae = 300$

$|AP - BP| = 2a = 200 \ \Rightarrow \ a = 100$

$b^2 = (ae)^2 - a^2 = 300^2 - 100^2$

$= 90000 - 10000 \ = 80000$

$\therefore$ Equation $\frac{(y-300)^2}{10000} - \frac{x^2}{80000} = 1$.

---

10) Points A and B are 10 km apart and it is determine from the sound of an explosion heard at those points at different times that the location of the explosion is 6 km closer to A than B . Show that the location of the explosion is restricted to a particular curve and find an equation of if

SOLUTION:

Let A and B be the focus.

$AB = 2ae = 10 \Rightarrow \quad ae = 5$

Let p be the point of explosion.

$|AP - BP| = 2a = 6$

$\therefore \ b^2 = (ae)^2 - a^2 = 5^2 - 3^2 = 25 - 9 = 16$

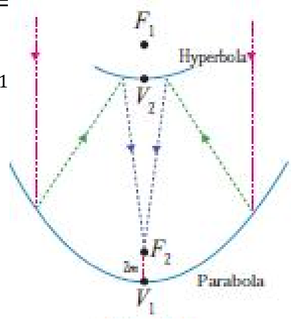
Locus of pt p is Hyperbola center (0,0)

$\frac{x^2}{9} - \frac{y^2}{16} = 1$

EXAMPLE 5.35: Certain telescopes contain both parabolic mirror and a hyperbolic mirror. In the telescope shown in figure the parabola and hyperbola share focus F_1 which is 14 m above the vertex of the parabola. The hyperbola's second focus F_2 is 2m above the parabola's vertex . the vertex of the hyperbolic mirror is 1 m below F_1 . Position of coordinate system with the origin af the centre of the hyperbola and with the foci on the y -axis Then find the equation of the hyperbola.

Solution :

$V_1 =$ vertex of parabola & $V_2 =$ Vertex of hyperbola



F_1& F_2 are Foci of Hyperbola but F_1 is focus of parabola also

$V_1F_1 = 14$ m $\quad V_1F_2 = 2$m

$F_1F_2 = 2ae = 14 - 2 = 12$m

$CF_1 = ae = 6$m

$a = 6 - 1 = 5$m $\Rightarrow a^2 = 25$

$b^2 = (ae)^2 - a^2 = 6^2 - 5^2$

$b^2 = 36 - 25 = 11$

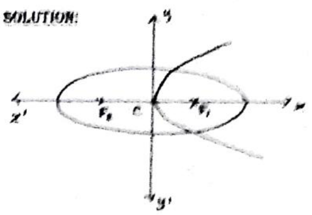
Transverse axis y axis center (0,0).

$\therefore \frac{y^2}{25} - \frac{x^2}{11} = 1$

EXAMPLE 5.36: An equation of the elliptical part of an optical lens system is $\frac{x^2}{16} + \frac{y^2}{9} = 1$. The parabolic part of the system has a focus in common with the right focus of the ellipse .The vertex of the parabola is at the origin and the parabola opens to the right. Determine the equation of the parabola.

Solution :

$\frac{x^2}{16} + \frac{y^2}{9} \quad a^2 = 16$



$b^2 = 9ae = \sqrt{a^2 - b^2}$

$= \sqrt{16 - 9}$

Foci of ellipse are

$(\sqrt{7}, 0), (-\sqrt{7}, 0)$

Given parabolic part of focus

coincides with right focus of ellipse parabola opens right.

$\therefore$ Eqn is $y^2 = 4ax$

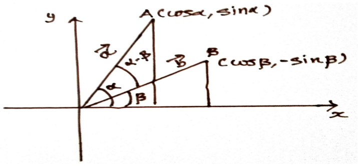
$\therefore y^2 = 4\sqrt{7}x$

CHAPTER 6 - VECTOR ALGEBRA
(5 MARKS ONLY)

EXERCISE 6.1 (5): Prove by vector method:

$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

Solution:



$|\vec{a}| = |\vec{b}| = 1$

$\angle AOB = \alpha - \beta$

$A(\cos\alpha, \sin\alpha)$

$B(\cos\beta, \sin\beta)$

$\vec{a} = \cos\alpha\,\hat{i} + \sin\alpha\,\hat{j}$ & $\vec{b} = \cos\beta\,\hat{i} + \sin\beta\,\hat{j}$

$\vec{b}.\vec{a} = |\vec{b}||\vec{a}|\cos(\alpha - \beta) = (1)(1)\cos(\alpha - \beta)$

$= \cos(\alpha - \beta)$ ______(1)

$\vec{b}.\vec{a} = (\cos\beta\,\hat{i} + \sin\beta\,\hat{j}).(\cos\alpha\,\hat{i} + \sin\alpha\,\hat{j})$

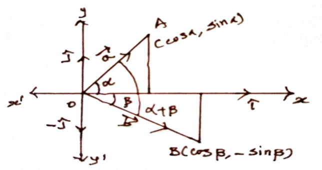
$= \cos\alpha\cos\beta + \sin\alpha\sin\beta$ ______(2)

From (1) and (2) $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

EXAMPLE 6.3 : Prove by vector method:

$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ Solution:

Solution:



$|\vec{a}| = |\vec{b}| = 1$ &

$\angle AOB = \alpha + \beta$

$A(\cos\alpha, \sin\alpha)$ &

$B(\cos\beta, -\sin\beta)$

$\vec{a} = \cos\alpha\,\hat{i} + \sin\alpha\,\hat{j}$ & $\vec{b} = \cos\beta\,\hat{i} - \sin\beta\,\hat{j}$

$\vec{b}.\vec{a} = |\vec{b}||\vec{a}|\cos(\alpha + \beta) = (1)(1)\cos(\alpha + \beta)$

$= \cos(\alpha + \beta)$ ______(1)

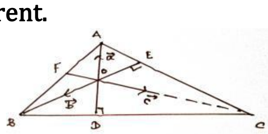
$\vec{b}.\vec{a} = (\cos\beta\,\hat{i} - \sin\beta\,\hat{j}).(\cos\alpha\,\hat{i} + \sin\alpha\,\hat{j})$

$= \cos\alpha\cos\beta - \sin\alpha\sin\beta$ ______(2)

From (1) and (2) $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

EXAMPLE 6.7 : Prove by vector method that the perpendiculars (ALTITUDES) drawn from the vertices to the opposite sides of a triangle are concurrent.

Solution:



In triangle ABC,
Altitudes AD, BE meet at O.
To prove the third altitude from c to AB also pass through O.

$AD \perp BC \Rightarrow OA \perp BC$	$BE \perp CA \Rightarrow OB \perp CA$
$\Rightarrow \overrightarrow{OA} \perp \overrightarrow{BC}$	$\Rightarrow \overrightarrow{OB} \perp \overrightarrow{CA}$
$\Rightarrow \overrightarrow{OA}\,.\,\overrightarrow{BC} = 0$	$\Rightarrow \overrightarrow{OB}\,.\,\overrightarrow{CA} = 0$
$\Rightarrow \overrightarrow{OA}.(\overrightarrow{OC} - \overrightarrow{OB}) = 0$	$\Rightarrow \overrightarrow{OB}.(\overrightarrow{OA} - \overrightarrow{OC}) = 0$
$\Rightarrow \overrightarrow{OA}.\overrightarrow{OC} - \overrightarrow{OA}.\overrightarrow{OB} = 0$	$\Rightarrow \overrightarrow{OB}.\overrightarrow{OA} - \overrightarrow{OB}.\overrightarrow{OC} = 0$
$\Rightarrow \overrightarrow{OA}.\overrightarrow{OC} = \overrightarrow{OA}.\overrightarrow{OB}$	$\Rightarrow \overrightarrow{OB}.\overrightarrow{OA} = \overrightarrow{OB}.\overrightarrow{OC}$

$\overrightarrow{OA}.\overrightarrow{OC} = \overrightarrow{OA}.\overrightarrow{OB}$ & $\overrightarrow{OB}.\overrightarrow{OA} = \overrightarrow{OB}.\overrightarrow{OC}$

$\Rightarrow \overrightarrow{OA}.\overrightarrow{OC} = \overrightarrow{OB}.\overrightarrow{OC}$

$\Rightarrow \overrightarrow{OC}.\overrightarrow{OA} = \overrightarrow{OC}.\overrightarrow{OB}$

$\Rightarrow \overrightarrow{OC}.\overrightarrow{OB} - \overrightarrow{OC}.\overrightarrow{OA} = 0$

$\Rightarrow \overrightarrow{OC}.(\overrightarrow{OB} - \overrightarrow{OA}) = 0 \quad \Rightarrow \overrightarrow{OC}.\overrightarrow{AB} = 0$

$\Rightarrow \overrightarrow{OC} \perp \overrightarrow{AB}$

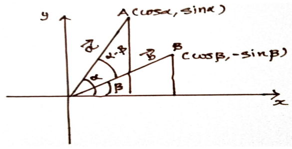
$\Rightarrow OC \perp AB$

Altitude from C to AB also pass through O

EXAMPLE 6.5: Prove by vector method:

$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

Solution:



$|\vec{a}| = |\vec{b}| = 1$

$\angle AOB = \alpha - \beta$

$A(\cos\alpha, \sin\alpha)$

$B(\cos\beta, \sin\beta)$

$\vec{a} = \cos\alpha\,\hat{i} + \sin\alpha\,\hat{j}$ & $\vec{b} = \cos\beta\,\hat{i} + \sin\beta\,\hat{j}$

$\vec{b} \times \vec{a} = |\vec{b}||\vec{a}|\sin(\alpha - \beta)\,\hat{k} = (1)(1)\sin(\alpha - \beta)\,\hat{k}$

$= \sin(\alpha - \beta)\,\hat{k}$ ______(1)

$$\vec{b} \times \vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\beta & \sin\beta & 0 \\ \cos\alpha & \sin\alpha & 0 \end{vmatrix}$$

$= \hat{i}(0) - \hat{j}(0) + \hat{k}(\cos\beta\sin\alpha - \cos\alpha\sin\beta)$

$= \hat{k}(\cos\beta\sin\alpha - \cos\alpha\sin\beta)$ ______(2)

From (1) and (2)

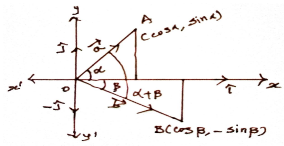
$\sin(\alpha - \beta)\,\hat{k} = \hat{k}(\cos\beta\sin\alpha - \cos\alpha\sin\beta)$

$\sin(\alpha - \beta) = (\cos\beta\sin\alpha - \cos\alpha\sin\beta)$

EXERCISE 6.1(10): Prove by vector method:-

$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

Solution:



$|\vec{a}| = |\vec{b}| = 1$

$\angle AOB = \alpha - \beta$

$A(\cos\alpha, \sin\alpha)$ & $B(\cos\beta, -\sin\beta)$

$\vec{a} = \cos\alpha\,\hat{i} + \sin\alpha\,\hat{j}$ & $\vec{b} = \cos\beta\,\hat{i} - \sin\beta\,\hat{j}$

$\vec{b} \times \vec{a} = |\vec{b}||\vec{a}|\sin(\alpha + \beta)\,\hat{k} = (1)(1)\sin(\alpha + \beta)\,\hat{k}$

$= \sin(\alpha + \beta)\,\hat{k}$ ______(1)

$$\vec{b} \times \vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\beta & -\sin\beta & 0 \\ \cos\alpha & \sin\alpha & 0 \end{vmatrix}$$

$= \hat{i}(0) - \hat{j}(0) + \hat{k}(\cos\beta\sin\alpha + \cos\alpha\sin\beta)$

$= \hat{k}(\cos\beta\sin\alpha + \cos\alpha\sin\beta)$ ______(2)

From (1) and (2)

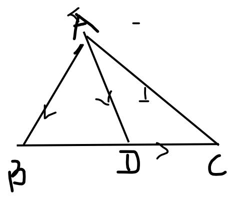
$\sin(\alpha + \beta)\,\hat{k} = \hat{k}(\cos\beta\sin\alpha + \cos\alpha\sin\beta)$

$\sin(\alpha + \beta) = (\cos\beta\sin\alpha + \cos\alpha\sin\beta)$

Example 6.6: If D is the midpoint of the side BC of a triangle ABC, show by vector method that

$|\overrightarrow{AB}|^2 + |\overrightarrow{AC}|^2 = 2(|\overrightarrow{AD}|^2 + |\overrightarrow{BD}|^2)$

Solution:



In triangle ABC , D is mid point of BC

$BD = DC$ & $\overrightarrow{DB} = -\overrightarrow{DC}$

Equal magnitude but opposite direction

$|\overrightarrow{AB}|^2 + |\overrightarrow{AC}|^2$

$= |\overrightarrow{AB} + \overrightarrow{DB}|^2 + |\overrightarrow{AB} + \overrightarrow{DC}|^2 \qquad \overrightarrow{DC} = -\overrightarrow{DB}$

$= |\overrightarrow{AD} + \overrightarrow{DB}|^2 + |\overrightarrow{AD} - \overrightarrow{DB}|^2$

$= |\overrightarrow{AD}|^2 + |\overrightarrow{DB}|^2 + 2\overrightarrow{AD}\cdot\overrightarrow{DB} + |\overrightarrow{AD}|^2 + |\overrightarrow{DB}|^2 - 2\overrightarrow{AD}\cdot\overrightarrow{DB}$

$= 2|\overrightarrow{AD}|^2 + 2|\overrightarrow{DB}|^2$

$= 2(|\overrightarrow{AD}|^2 + |\overrightarrow{DB}|^2)$

$= 2(|\overrightarrow{AD}|^2 + |\overrightarrow{BD}|^2)$

Exercise 6.3 (4):

If $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}, \vec{b} = 3\hat{i} + 5\hat{j} + 2\hat{k}, \vec{c} = -\hat{i} - 2\hat{j} + 3\hat{k}$, verify that

(i) $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a}.\vec{c})\vec{b} - (\vec{b}.\vec{c})\vec{a}$

(ii) $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b}) \times \vec{c}$

Solution:

$\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}, \vec{b} = 3\hat{i} + 5\hat{j} + 2\hat{k}, \vec{c} = -\hat{i} - 2\hat{j} + 3\hat{k}$

(i) $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a}.\vec{c})\vec{b} - (\vec{b}.\vec{c})\vec{a}$

L.H.S

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & -1 \\ 3 & 5 & 2 \end{vmatrix} = \hat{i}(6+5) - \hat{j}(4+3) + \hat{k}(10-9)$$

$$= \hat{i}(11) - \hat{j}(7) + \hat{k}(1) = 11\hat{i} - 7\hat{j} + \hat{k}$$

$$(\vec{a} \times \vec{b}) \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 11 & -7 & 1 \\ -1 & -2 & 3 \end{vmatrix}$$

$$= \hat{i}(-21+2) - \hat{j}(33+1) + \hat{k}(-22-7)$$

$$= \hat{i}(-19) - \hat{j}(34) + \hat{k}(-29) = -19\hat{i} - 34\hat{j} - 29\hat{k}.$$

R.H.S. $\vec{a}.\vec{c} = (2\hat{i} + 3\hat{j} - \hat{k}).(-\hat{i} - 2\hat{j} + 3\hat{k})$

$$= 2(-1) + 3(-2) + (-1)(3) = -2-6-3 = -11$$

$\vec{b}.\vec{c} = (3\hat{i} + 5\hat{j} + 2\hat{k}).(-\hat{i} - 2\hat{j} + 3\hat{k})$

$$= 3(-1) + 5(-2) + 2(3) = -3-10+6 = -13+6 = -7$$

$(\vec{a}.\vec{c})\vec{b} - (\vec{b}.\vec{c})\vec{a}$

$= -11(3\hat{i} + 5\hat{j} + 2\hat{k}) - (-7)(2\hat{i} + 3\hat{j} - \hat{k})$

$= -33\hat{i} - 55\hat{j} - 22\hat{k} + 14\hat{i} + 21\hat{j} - 7\hat{k} = -19\hat{i} - 34\hat{j} - 29\hat{k}$

L. H.S = R.H.S $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a}.\vec{c})\vec{b} - (\vec{b}.\vec{c})\vec{a}$

(ii) $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$

L.H.S

$$\vec{b} \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 5 & 2 \\ -1 & -2 & 3 \end{vmatrix} = \hat{i}(15+4) - \hat{j}(9+2) + \hat{k}(-6+5)$$

$$= \hat{i}(19) - \hat{j}(11) + \hat{k}(-1) = 19\hat{i} - 11\hat{j} - 1\hat{k}.$$

$$\vec{a} \times (\vec{b} \times \vec{c}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & -1 \\ 19 & -11 & -1 \end{vmatrix}$$

$$= \hat{i}(-3-11) - \hat{j}(-2+19) + \hat{k}(-22-57)$$

$$= \hat{i}(-14) - \hat{j}(17) + \hat{k}(-79) = -14\hat{i} - 17\hat{j} - 79\hat{k}.$$

$\vec{a}.\vec{c} = (2\hat{i} + 3\hat{j} - \hat{k}).(-\hat{i} - 2\hat{j} + 3\hat{k})$

$$= 2(-1) + 3(-2) + (-1)(3) = -2-6-3 = -11$$

$\vec{a}.\vec{b} = (2\hat{i} + 3\hat{j} - \hat{k}).(3\hat{i} + 5\hat{j} + 2\hat{k})$

$$= 2(3) + 3(5) + (-1)(2) = 6+15-2 = 19$$

$(\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$

$= (-11)(3\hat{i} + 5\hat{j} + 2\hat{k}) - 19(-\hat{i} - 2\hat{j} + 3\hat{k})$

$= -33\hat{i} - 55\hat{j} - 22\hat{k} + 19\hat{i} + 38\hat{j} - 57\hat{k} = -14\hat{i} - 17\hat{j} - 79\hat{k}$

$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$

Example 6.23:

If $\vec{a} = \hat{i} - \hat{j}, \vec{b} = \hat{i} - \hat{j} - 4\hat{k}, \vec{c} = 3\hat{j} - \hat{k}, \vec{d} = 2\hat{i} + 5\hat{j} + \hat{k}$, verify that

(i) $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a}, \vec{b}, \vec{d}]\vec{c} - [\vec{a}, \vec{b}, \vec{c}]\vec{d}$

Solution:

$\vec{a} = \hat{i} - \hat{j}, \vec{b} = \hat{i} - \hat{j} - 4\hat{k}, \vec{c} = 3\hat{j} - \hat{k}, \vec{d} = 2\hat{i} + 5\hat{j} + \hat{k}$

(i) $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a}, \vec{b}, \vec{d}]\vec{c} - [\vec{a}, \vec{b}, \vec{c}]\vec{d}$

L.H.S

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 0 \\ 1 & -1 & -4 \end{vmatrix} \quad \begin{aligned} &= \hat{i}(4-0) - \hat{j}(-4-0) + \hat{k}(-1+1) \\ &= \hat{i}(4) - \hat{j}(-4) + \hat{k}(0) = 4\hat{i} + 4\hat{j} \end{aligned}$$

$$\vec{c} \times \vec{d} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 3 & -1 \\ 2 & 5 & 1 \end{vmatrix} \quad \begin{aligned} &= \hat{i}(3+5) - \hat{j}(0+2) + \hat{k}(0-6) \\ &= \hat{i}(8) - \hat{j}(2) + \hat{k}(-6) = 8\hat{i} - 2\hat{j} - 6\hat{k} \end{aligned}$$

$$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 4 & 0 \\ 8 & -2 & -6 \end{vmatrix}$$

$$= \hat{i}(-24-0) - \hat{j}(-24-0) + \hat{k}(-8-32)$$

$$= \hat{i}(-24) - \hat{j}(-24) + \hat{k}(-40) = -24\hat{i} + 24\hat{j} - 40\hat{k}$$

R.H.S

$$[\vec{a}, \vec{b}, \vec{d}] = \begin{vmatrix} 1 & -1 & 0 \\ 1 & -1 & -4 \\ 2 & 5 & 1 \end{vmatrix} = 1(-1+20) + 1(1+8) + 0(5+2)$$

$$= 1(19) + 1(9) + 0 = 19+9 = 28$$

$$[\vec{a}, \vec{b}, \vec{c}] = \begin{vmatrix} 1 & -1 & 0 \\ 1 & -1 & -4 \\ 0 & 3 & -1 \end{vmatrix} = 1(1+12) + 1(-1-0) + 0(3+0)$$

$$= 1(13) + 1(-1) + 0 = 13 - 1 = 12$$

$[\vec{a}, \vec{b}, \vec{d}]\vec{c} - [\vec{a}, \vec{b}, \vec{c}]\vec{d} = 28(3\hat{j} - \hat{k}) - 12(2\hat{i} + 5\hat{j} + \hat{k})$

$$= 84\hat{j} - 28\hat{k} - 24\hat{i} - 60\hat{j} - 12\hat{k} = -24\hat{i} + 24\hat{j} - 40\hat{k}$$

L.H.S=R.H.S $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a}, \vec{b}, \vec{d}]\vec{c} - [\vec{a}, \vec{b}, \vec{c}]\vec{d}$

(ii) $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a}, \vec{c}, \vec{d}]\vec{b} - [\vec{b}, \vec{c}, \vec{d}]\vec{a}$

L.H.S:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 0 \\ 1 & -1 & -4 \end{vmatrix} \quad \begin{aligned} &= \hat{i}(4-0) - \hat{j}(-4-0) + \hat{k}(-1+1) \\ &= \hat{i}(4) - \hat{j}(-4) + \hat{k}(0) = 4\hat{i} + 4\hat{j} \end{aligned}$$

$$\vec{c} \times \vec{d} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 3 & -1 \\ 2 & 5 & 1 \end{vmatrix} \quad \begin{aligned} &= \hat{i}(3+5) - \hat{j}(0+2) + \hat{k}(0-6) \\ &= \hat{i}(8) - \hat{j}(2) + \hat{k}(-6) = 8\hat{i} - 2\hat{j} - 6\hat{k} \end{aligned}$$

$$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 4 & 0 \\ 8 & -2 & -6 \end{vmatrix}$$

$$= \hat{i}(-24-0) - \hat{j}(-24-0) + \hat{k}(-8-32)$$

$$= \hat{i}(-24) - \hat{j}(-24) + \hat{k}(-40) = -24\hat{i} + 24\hat{j} - 40\hat{k}$$

$$[\vec{a}, \vec{c}, \vec{d}] = \begin{vmatrix} 1 & -1 & 0 \\ 0 & 3 & -1 \\ 2 & 5 & 1 \end{vmatrix} \quad \begin{aligned} &= 1(3+5) + 1(0+2) + 0(0-6) \\ &= 1(8) + 1(2) + 0 = 8+2 = 10 \end{aligned}$$

$$[\vec{b}, \vec{c}, \vec{d}] = \begin{vmatrix} 1 & -1 & -4 \\ 0 & 3 & -1 \\ 2 & 5 & 1 \end{vmatrix} \quad \begin{aligned} &= 1(3+5) + 1(0+2) - 4(0-6) \\ &= 1(8) + 1(2) - 4(-6) \\ &= 8+2+24 = 34 \end{aligned}$$

$[\vec{a}, \vec{c}, \vec{d}]\vec{b} - [\vec{b}, \vec{c}, \vec{d}]\vec{a}$

$= 10(\hat{i} - \hat{j} - 4\hat{k}) - (34)(\hat{i} - \hat{j}) = 10((\hat{i} - \hat{j} - 4\hat{k}) - 34(\hat{i} - \hat{j})$

$= 10\hat{i} - 10\hat{j} - 40\hat{k} - 34\hat{i} + 34\hat{j} = -24\hat{i} + 24\hat{j} - 40\hat{k}$

L.H.S = R.H.S

Exercise 6.5(4):

Show that the lines $\frac{x-3}{3}=\frac{y-3}{-1}$, z – 1 = 0.and $\frac{x-6}{2}=\frac{z-1}{3}$, y – 2 = 0 intersect. Also find the point of intersection.

Solution:

$\frac{x-3}{3}=\frac{y-3}{-1}, z-1=0 \Rightarrow \frac{x-3}{3}=\frac{y-3}{-1}= \frac{z-1}{0}$

$\frac{x-6}{2}=\frac{z-1}{3}, y-2=0 \Rightarrow \frac{x-6}{2}=\frac{y-2}{0}= \frac{z-1}{3}$

$\vec{a}= 3\hat{i}+3\hat{j}+\hat{k}$ $\quad\vec{b}= 3\hat{i}-\hat{j}+0\hat{k}$

$\vec{c}= 6\hat{i}+2\hat{j}+\hat{k}$ $\quad\vec{d}=2\hat{i}+0\hat{j}+3\hat{k}$

$\vec{b}$ and $\vec{d}$ are not parallel.

$\vec{c}-\vec{a}= 6\hat{i}+2\hat{j}+\vec{k}-3\hat{i}-3\hat{j}-\vec{k}=3\hat{i}-\hat{j}$

$\vec{b}\times\vec{d}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\3&-1&0\\2&0&3\end{vmatrix}=\hat{i}(-3)-\hat{j}(9)+\hat{k}(0+2)$

$=-3\hat{i}-9\hat{j}+2\hat{k}$

$(\vec{c}-\vec{a}).(\vec{b}\times\vec{d})=(3\hat{i}-\hat{j}).(-3\hat{i}-9\hat{j}+2\hat{k})$

$=3(-3)+(-1)(-9)+0$

$=0$

The lines are intersecting

Any point on the line

$\frac{x-3}{3}= \frac{y-3}{-1}= \frac{z-1}{0}=\lambda$

$\frac{x-3}{3}=\lambda, \frac{y-3}{-1}=\lambda, \frac{z-1}{0}=\lambda$

$x-3=3\lambda, y-3=-\lambda, z-1=0$

$x=3+3\lambda, y=3-\lambda, z=1$

Any point $(3+3\lambda, 3-\lambda, 1)$

$\frac{x-6}{2}= \frac{y-2}{0}= \frac{z-1}{3}=\mu$

$\frac{x-6}{2}=\mu, \frac{y-2}{0}=\mu, \frac{z-1}{3}=\mu$

$x-6=2\mu, y-2=0, z-1=3\mu$

$x-6=2\mu, y-2=0, z-1=3\mu$

$x=2\mu+6$, y

$=2, z=3\mu+1$

any point $(2\mu+6, 2, 3\mu+1)$

Since line intersects for some λ and μ

$(3+3\lambda, 3-\lambda, 1)=(2\mu+6, 2, 3\mu+1)$

$3-\lambda=2 \Rightarrow -\lambda=2-3=-1 \Rightarrow \lambda=1$

$3\mu+1=1 \Rightarrow 3\mu=1-1=0, \Rightarrow \mu=0.$

Point of intersection (6, 2, 1)

Example 6.33: **Show that the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $\frac{x-4}{5}=\frac{y-1}{2}=z$ intersect. Also find the point of intersection.**

Solution:

$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ $\quad\frac{x-4}{5}=\frac{y-1}{2}=z=\frac{z-0}{1}$

$\vec{a}=\hat{i}+2\hat{j}+3\hat{k}$ $\quad\vec{b}=2\hat{i}+3\vec{j}+4\hat{k}$

$\vec{c}=4\hat{i}+\hat{j}+0\hat{k}$ $\quad\vec{d}=5\hat{i}+2\hat{j}+\hat{k}$

$\vec{b}$ and $\vec{d}$ are not parallel.

$\vec{c}-\vec{a}=4\hat{i}+\hat{j}+0\hat{k}-\hat{i}-2\hat{j}-3\hat{k}=3\hat{i}-\hat{j}-3\hat{k}$

$\vec{b}\times\vec{d}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\2&3&4\\5&2&1\end{vmatrix}=\hat{i}(3-8)-\hat{j}(2-10)+\hat{k}(4-15)$

$=-5\hat{i}+8\hat{j}-11\hat{k}$

$(\vec{c}-\vec{a}).(\vec{b}\times\vec{d})=(3\hat{i}-\hat{j}-3\hat{k}).(-5\hat{i}+8\hat{j}-11\hat{k})$

$=3(-5)+(-1)(8)+(-3)(-11)$

$=-15-8+33=0$

The lines are intersecting

Any point on the line

$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}=\lambda$

$\frac{x-1}{2}=\lambda, \frac{y-2}{3}=\lambda, \frac{z-3}{4}=\lambda$

$x-1=2\lambda, y-2=3\lambda, z-3=4\lambda$

$x=2\lambda+1, y=3\lambda+2, z=4\lambda+3$

Any point $(2\lambda+1, 3\lambda+2, 4\lambda+3)$

Any point on the second line

$\frac{x-4}{5}=\frac{y-1}{2}=\frac{z-0}{1}=\mu$

$\frac{x-4}{5}=\mu, \frac{y-1}{2}=\mu, \frac{z-0}{1}=\mu$

$x-4=5\mu, y-1=2\mu, z=\mu$

$x=5\mu+4, y=2\mu+1, z=\mu$

Any point $(5\mu+4, 2\mu+1, \mu)$

Since line intersects for some λ and μ

$(2\lambda+1, 3\lambda+2, 4\lambda+3)=(5\mu+4, 2\mu+1, \mu)$

Equating x–co ordinate $2\lambda+1=5\mu+4$

$\Rightarrow 2\lambda-5\mu=3$ ______ (1)

Equating z-coordinate: $4\lambda+3=\mu$

$\Rightarrow 4\lambda-\mu=-3$ ______(2)

(1)x2 $\Rightarrow 4\lambda-10\mu= 6$
(2)x1 $\Rightarrow 4\lambda-\mu=-3$
– + +

$-9\mu=9$
$\mu=-1$

Substitute $\mu=-1$ in $4\lambda-\mu=-3$
$4\lambda-(-1)=-3$
$4\lambda+1=-3$
$4\lambda=-3-1=-4$
$\lambda=-1$

So point of intersection is (-1, -1, -1)

Example 6.37: **Find the coordinate of the perpendicular drawn from the point (-1, 2, 3) to the straight line $\vec{r} = (\hat{i} - 4\hat{j} + 3\hat{k}) + t(2\hat{i} + 3\hat{j} + \hat{k})$, also find the shortest distance from the given point to the straight line.**

Solution: $\vec{r} = (\hat{i} - 4\hat{j} + 3\hat{k}) + t(2\hat{i} + 3\hat{j} + \hat{k})$

$\vec{a} = \hat{i} - 4\hat{j} + 3\hat{k}$ $\quad (x_1, y_1, z_1) = (1, -4, 3)$

$\vec{b} = 2\hat{i} + 3\hat{j} + \hat{k}$ $\quad (b_1, b_2, b_3) = (2, 3, 1)$

Cartesian equation is : $\frac{x - x_1}{b_1} = \frac{y - y_1}{b_2} = \frac{z - z_1}{b_3}$

$\frac{x-1}{2} = \frac{y+4}{3} = \frac{z-3}{1}$

To find any point : $\frac{x-1}{2} = \frac{y+4}{3} = \frac{z-3}{1} = t$

$\frac{x-1}{2} = t, \frac{y+4}{3} = t, \frac{z-3}{1} = t$

$x = 2t + 1, y = 3t - 4, z = t + 3$

Any point is (2t +1, 3t - 4, t + 3)

Let foot of the perpendicular B(1 + 32t +1, 3t - 4, t + 3)

Point A(-1, 2, 3)

Direction ratios of line joining two points A and B

D.r's = (2t + 1 + 1, 3t - 4 – 2, t + 3 – 3)= (2t + 2, 3t - 6, t)

D.r's 0f the given line is 2, 3,1

Since lines are perpendicular:

$2(2t + 2)+3(3t - 6)+(1)(t) = 0$

$4t + 4 + 9t – 18 + t = 0 \Rightarrow 14t – 14 = 0 \Rightarrow 14t = 14 \Rightarrow t = 1$

Point of intersection is (2(1)+1, 3(1) -4, 1 + 3)

= (3, -1, 4)

Shortest distance of the point A from the line

A = (-1, 2, 3) and B(3, -1, 4)

$AB = \sqrt{(3 - (-1))^2 + (-1 - 2)^2 + (4 - 3)^2}$

$= \sqrt{(3 + 1)^2 + (-3)^2 + 1^2} = \sqrt{16 + 9 + 1} = \sqrt{26}$

Example 6.35: **Determine whether the pair of straight lines $\vec{r} = (2\hat{i} + 6\hat{j} + 3\hat{k}) + t(2\hat{i} + 3\hat{j} + 4\hat{k})$ and $\vec{r} = (2\hat{j} - 3\hat{k}) + s(\hat{i} + 2\hat{j} + 3\hat{k})$ are parallel and find the shortest distance between them**

SOLUTION;

$\vec{a} = 2\hat{i} + 6\hat{j} + 3\hat{k}$ $\quad \vec{b} = 2\hat{i} + 3\hat{j} + 4\hat{k}$

$\vec{c} = 2\hat{j} - 3\hat{k}$ $\quad \vec{d} = \hat{i} + 2\hat{j} + 3\hat{k}$

clearly $\vec{b}$ is not scalar multiple of $\vec{d}$ so the vectors are not parallel and hence the lines are not parallel.

$\vec{c} - \vec{a} = 2\hat{j} - 3\hat{k} - 2\hat{i} - 6\hat{j} - 3\hat{k} = -2\hat{i} - 4\hat{j} - 6\hat{k}$

$\vec{b} \times \vec{d} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 1 & 2 & 3 \end{vmatrix} = \hat{i}(9 - 8) - \hat{j}(6 - 4) + \hat{k}(4 - 3)$

$= \hat{i} - 2\hat{j} + \hat{k}$

$(\vec{c} - \vec{a}).(\vec{b} \times \vec{d}) = (-2\hat{i} - 4\hat{j} - 6\hat{k}).(\hat{i} - 2\hat{j} + \hat{k})$

$= (-2)(1) + (-4)(-2) + (-6)(1)$

$= -2 + 8 - 6 = 0$

The lines are coplanar so lines are intersecting

so distance = 0

Example 6.34:

Find the parametric form of vector equation of a straight line passing through the point of intersection of the straight lines $\vec{r} = (\hat{i} + 3\hat{j} - \hat{k}) + t(2\hat{i} + 3\hat{j} + 2\hat{k})$ And $\frac{x-2}{1} = \frac{y-4}{2} = \frac{z+3}{4}$ and perpendicular to both straight lines.

Solution:

$\vec{r} = (\hat{i} + 3\hat{j} - \hat{k}) + t(2\hat{i} + 3\hat{j} + 2\hat{k})$

$\vec{a} = \hat{i} + 3\hat{j} - \hat{k}$ $\quad (x_1, y_1, z_1) = (1, 3, -1)$

$\vec{b} = 2\hat{i} + 3\hat{j} + 2\hat{k}$ $\quad (b_1, b_2, b_3) = (2, 3, 2)$

Cartesian equation is : $\frac{x - x_1}{b_1} = \frac{y - y_1}{b_2} = \frac{z - z_1}{b_3}$

$\frac{x-1}{2} = \frac{y-3}{3} = \frac{z+1}{2}$

To find any point : $\frac{x-1}{2} = \frac{y-3}{3} = \frac{z+1}{2} = t$

$\frac{x-1}{2} = t, \frac{y-3}{3} = t, \frac{z+1}{2} = t$

$x = 2t + 1, y = 3t + 3, z = 2t – 1$

Any point is (2t +1, 3t + 3, 2t – 1)

SECOND LINE: $\frac{x-2}{2} = \frac{y-4}{2} = \frac{z+3}{4}$

To find any point let $\frac{x-2}{1} = \frac{y-4}{2} = \frac{z+3}{4} = s$

$\frac{x-2}{1} = s, \frac{y-4}{2} = s, \frac{z+3}{4} = s$

$x = s + 2, y = 2s + 4, z = 4s - 3$

Any point (s + 2, 2s + 4, 4s – 3)

Since lines are intersecting

(2t +1, 3t + 3, 2t – 1) = (s + 2, 2s + 4, 4s – 3)

x coordinate : $2t + 1 = s + 2 \Rightarrow 2t – s = 2-1 \Rightarrow 2t – s = 1$

y coordinate: $3t + 3 = 2s + 4 \Rightarrow 3t – 2s = 4 – 3 \Rightarrow 3t – 2s = 1$

z coordinate : $2t – 1 = 4s – 3 \Rightarrow 2t – 4s = -3+1 \Rightarrow$

$2t - 4s = -2$

$2t - 4s = -2$ divide by 2 $\Rightarrow t – 2s = -1$

solving we get t = 1 and s = 1

point is (1+2, 2(1)+4, 4(1)-3) = (3, 6, 1)

$\vec{b} \times \vec{d} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 2 \\ 1 & 2 & 4 \end{vmatrix} = \hat{i}(12 – 4) - \hat{j}(8 - 2) + \hat{k}(4 - 3)$

$= 8\hat{i} - 6\hat{j} + \hat{k}$

Equation of line: through (3,6,1) and parallel to $8\hat{i} - 6\hat{j} + \hat{k}$

$\vec{a} = 3\hat{i} + 6\hat{j} + \hat{k}$ $\vec{b} = \hat{i} - 6\hat{j} + \hat{k}$

Vector equation: $\vec{r} = \vec{a} + t\vec{b}, \quad t \in \mathbb{R}$

$\vec{a} = (3\hat{i} + 6\hat{j} + \hat{k}) + t(\hat{i} - 6\hat{j} + \hat{k})$

Example 6.43

Find the non-parametric form of vector equation , and Cartesian equation of the plane passing through the point (0,1,-5) and parallel to the straight lines

$\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + s(2\hat{i} + 3\hat{j} + 6\hat{k})$ **and**

$\vec{r} = (\hat{i} - 3\hat{j} + 5\hat{k}) + t(\hat{i} + \hat{j} - \hat{k})$ **.**

Solution:

Point : $\vec{a} = 0\hat{i} + 1\hat{j} - 5\hat{k}$ $(x_1, y_1, z_1) = (0,1,-5)$

Parallel vector : $\vec{b} = 2\hat{i} + 3\hat{j} + 6\hat{k}$ $(b_1, b_2, b_3) = (2, 3, 6)$

Parallel vector: $\vec{c} = \hat{i} + \hat{j} - \hat{k}$ $(c_1, c_2, c_3) = (1,1,-1)$

Cartesian Equation : $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$

$\begin{vmatrix} x - 0 & y - 1 & z + 5 \\ 2 & 3 & 6 \\ 1 & 1 & -1 \end{vmatrix} = 0$

$\Rightarrow(x)(-3-6) - (y-1)(-2-6)+(z+5)(2-3) = 0$

$\Rightarrow x(-9) - (y-1)(-8)+(z+5)(-1) = 0$

$\Rightarrow -9x +8(y-1)-1(z+5) = 0$ $\Rightarrow -9x + 8y - 8 - z -5 = 0$

$\Rightarrow$ $-9x + 8y - z = 13$

$\Rightarrow 9x - 8y + z = -13$

Non parametric vector equation:

$(x\hat{i}+ y\hat{j} + z\hat{k}).(-9\hat{i} + 8\hat{j} - \hat{k}) = 13 \Rightarrow \vec{r}.(-9\hat{i} + 8\hat{j} - \hat{k}) = 13$

Example 6.44:

Find the vector parametric , vector non-parametric and Cartesian form of the equation of the plane passing through the points (-1,2,0) , (2,2,-1) and parallel to the straight line $\frac{x-1}{1} = \frac{2y+1}{2} = \frac{z+1}{-1}$.

Solution: $\frac{x-1}{1} = \frac{2y+1}{2} = \frac{z+1}{-1}$

rewritten as $\frac{x-1}{1} = \frac{2(y+\frac{1}{2})}{2} = \frac{z+1}{-1} \Rightarrow \frac{x-1}{1} = \frac{(y+\frac{1}{2})}{1} = \frac{z+1}{-1}$

Point : $\vec{a} = -1\hat{i} + 2\hat{j} + 0\hat{k}$ $(x_1, y_1, z_1) = (-1,2,0)$

Point : $\vec{b} = 2\hat{i} + 2\hat{j} - 1\hat{k}$ $(x_2, y_2, z_2) = (2, 2, -1)$

Parallel Vector :$\vec{c} = \hat{i} + \hat{j} - \hat{k}$ $(c_1, c_2, c_3) = (1, 1, -1)$

$\vec{b} - \vec{a} = 2\hat{i} + 2\hat{j} - 1\hat{k} - (-1\hat{i} + 2\hat{j} + 0\hat{k}) = 2\hat{i} + 2\hat{j} - 1\hat{k} + 1\hat{i} - 2\hat{j}$

$= 3\hat{i} + 0\hat{j} - 1\hat{k}$

Parametric Vector equation: $\vec{r} = \vec{a} + s(\vec{b} - \vec{a}) + t\vec{c}$

$\vec{r} = (-1\hat{i} + 2\hat{j} + 0\hat{k}) + s(3\hat{i} + 0\hat{j} - 1\hat{k}) + t(\hat{i} + \hat{j} - \hat{k})$

Cartesian Equation: $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$

$\begin{vmatrix} x - (-1) & y - 2 & z - 0 \\ 2 - (-1) & 2 - 2 & -1 - 0 \\ 1 & 1 & -1 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} x + 1 & y - 2 & z \\ 3 & 0 & -1 \\ 1 & 1 & -1 \end{vmatrix} = 0$

$\Rightarrow(x+1)(0+1)-(y-2)(-3+1)+(z)(3) = 0$

$\Rightarrow 1(x+1)+2(y-2)+3(z) = 0$ $\Rightarrow x +1+ 2y - 4+ 3z = 0$

$x + 2y + 3z - 3 = 0 \Rightarrow x + 2y + 3 = 3$

Non parametric vector equation:

$(x\hat{i}+ y\hat{j} + z\hat{k}).(\hat{i} + 2\hat{j}+3\hat{k}) = 3 \Rightarrow \vec{r}.(\hat{i} + 2\hat{j}+3\hat{k}) = 3$

Exercise 6.7(1)

Find the non-parametric form of vector equation and Cartesian equation of the plane passing through the point (2,3,6) and parallel to the straight lines $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-3}{1}$ **and** $\frac{x+3}{2} = \frac{y-3}{-5} = \frac{z+1}{-3}$.

Solution:

Point : $\vec{a} = 2\hat{i} + 3\hat{j} + 6\hat{k}$ $(x_1, y_1, z_1) = (2,3,6)$

Parallel vector : $\vec{b} = 2\hat{i} + 3\hat{j} + 1\hat{k}$ $(b_1, b_2, b_3) = (2, 3, 1)$

Parallel vector::$\vec{c} = 2\hat{i} - 5\hat{j} - 3\hat{k}$ $(c_1, c_2, c_3) = (2, -5, -3)$

Cartesian equation : $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$

$\begin{vmatrix} x - 2 & y - 3 & z - 6 \\ 2 & 3 & 1 \\ 2 & -5 & -3 \end{vmatrix} = 0$

$\Rightarrow(x-2)(-9+5) - (y-3)(-6-2)+(z-6)(-10-6) = 0$

$\Rightarrow(x-2)(-4) - (y - 3)(-8)+(z - 6)(-16) = 0$

$\Rightarrow -4(x-2) +8(y-3)-16(z - 6) = 0$

$\Rightarrow -4x + 8 + 8y - 24 - 16z + 96 = 0$

$\Rightarrow x - 2y + 4z + 80 = 0$ $\Rightarrow x - 2y + 4z = 20$

Non parametric vector equation:

$(x\hat{i}+ y\hat{j} + z\hat{k}).(\hat{i}- 2\hat{j}+4\hat{k}) = 20$ $\Rightarrow \vec{r}.(\hat{i}- 2\hat{j}+4\hat{k}) = 20$

Exercise 6.7 (2):

Find the parametric form of vector equation , and Cartesian equations of the plane passing through the points (2,2,1) , (9,3,6) and perpendicular to the plane $\mathbf{2x + 6y + 6z = 9}$.

Solution:

Point : $\vec{a} = 2\hat{i} + 2\hat{j} + \hat{k}$ $(x_1, y_1, z_1) = (2,2, 1)$

Point : $\vec{b} = 9\hat{i} + 3\hat{j} + 6\hat{k}$ $(x_2, y_2, z_2) = (9, 3, 6)$

Parallel Vector : $\vec{c} = 2\hat{i} + 6\hat{j} + 6\hat{k}$ $(c_1, c_2, c_3) = (2, 6, 6)$

$\vec{b} - \vec{a} = 9\hat{i} + 3\hat{j} + 6\hat{k} - (2\hat{i} + 2\hat{j} + \hat{k})$

$= 9\hat{i} + 3\hat{j} + 6\hat{k} - 2\hat{i} - 2\hat{j} - \hat{k} = 7\hat{i} + 1\hat{j} + 5\hat{k}$

Parametric Vector equation: $\vec{r} = \vec{a} + s(\vec{b} - \vec{a}) + t\vec{c}$

$\vec{r} = (2\hat{i} + 2\hat{j} + \hat{k}) + s(7\hat{i} + 1\hat{j} + 5\hat{k}) + t(2\hat{i} + 6\hat{j} + 6\hat{k})$

Cartesian Equation:: $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$

$\begin{vmatrix} x - 2 & y - 2 & z - 1 \\ 9 - 2 & 3 - 2 & 6 - 1 \\ 2 & 6 & 6 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} x - 2 & y - 2 & z - 1 \\ 7 & 1 & 5 \\ 2 & 6 & 6 \end{vmatrix} = 0$

$\Rightarrow(x- 2)(6 - 30) - (y - 2)(42 - 10) + (z - 1)(42 - 2) = 0$

$\Rightarrow -24(x - 2) - 32(y-2) + 40(z - 1) = 0$

$\Rightarrow -24x + 48 -32y + 64 +40z - 40 = 0$

$\Rightarrow -24x - 32y + 40z + 72 = 0$

$\Rightarrow 3x + 4y - 5z - 9 = 0$

Non parametric vector equation:

$(x\hat{i}+ y\hat{j} + z\hat{k}).(3\hat{i} +4\hat{j} - 5\hat{k}) - 9 = 0$

$\Rightarrow \vec{r}.(3\hat{i} + 4\hat{j} - 5\hat{k}) = 9$

Exercise 6.7(4)

Find the non-parametric form of vector equation and Cartesian equation of the plane passing through the point (1,-2,4) and perpendicular to the plane $x + 2y - 3z = 11$ and parallel to the line $\frac{x+7}{3} = \frac{y+3}{-1} = \frac{z}{1}$

Solution: $(1,-2,4)$ $\quad x + 2y - 3z = 11 \quad \frac{x+7}{3} = \frac{y+3}{-1} = \frac{z}{1}$

Point : $\vec{a} = \hat{i} - 2\hat{j} + 4\hat{k}$ $(x_1, y_1, z_1) = (1, -2, 4)$

Parallel Vector: $\vec{b} = \hat{i} + 2\hat{j} - 3\hat{k}$ $(b_1, b_2, b_3) = (1, 2, -3)$

Parallel Vector: $\vec{c} = 3\hat{i} - \hat{j} + \hat{k}$ $(c_1, c_2, c_3) = (3, -1, 1)$

Parametric Vector equation: $\vec{r} = \vec{a} + s\vec{b} + t\vec{c}$

$\Rightarrow \vec{r} = (\hat{i} - 2\hat{j} + 4\hat{k}) + s(\hat{i} + 2\hat{j} - 3\hat{k}) + t(3\hat{i} - \hat{j} + \hat{k})$

Cartesian Equation: $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$

$\begin{vmatrix} x - 1 & y + 2 & z - 4 \\ 1 & 2 & -3 \\ 3 & -1 & 1 \end{vmatrix} = 0$

$\Rightarrow (x - 1)(2-3) - (y+2)(1+9) + (z-4)(-1-6) = 0$

$\Rightarrow (x-1)(-1) - (y+2)(10) + (z-4)(-7) = 0$

$\Rightarrow -1(x - 1) - 10(y + 2) - 7(z - 4) = 0$

$\Rightarrow -x + 1 - 10y - 20 - 7z + 28 = 0 \quad \Rightarrow -x - 10y - 7z + 9 = 0$

$\Rightarrow x + 10y + 7z - 9 = 0$

Non Parametric Vector Equation:

$(x\hat{i} + y\hat{j} + z\hat{k}).(\hat{i} + 10\hat{j} + 7\hat{k}) - 9 = 0$

$\Rightarrow \vec{r}.(\hat{i} + 10\hat{j} + 7\hat{k}) - 9 = 0$

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3. Find parametric form of vector equation and Cartesian equations of the plane passing through the points (2,2,1) , (1,-2,3) and parallel to the straight line passing through the points (2,1,-3) and (-1,5,-8).

Solution: $\overrightarrow{OP} = 2\hat{i} + \hat{j} - 3\hat{k}$ $\quad \overrightarrow{OQ} = -\hat{i} + 5\hat{j} - 8\hat{k}$

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = -\hat{i} + 5\hat{j} - 8\hat{k} - 2\hat{i} - \hat{j} + 3\hat{k} = -3\hat{i} + 4\hat{j} - 5\hat{k}$

Point : $\vec{a} = 2\hat{i} + 2\hat{j} + \hat{k}$ $\quad (x_1, y_1, z_1) = (2, 2, 1)$

Point : $\vec{b} = \hat{i} - 2\hat{j} + 3\hat{k}$ $\quad (x_2, y_2, z_2) = (1, -2, 3)$

Parallel Vector : $\vec{c} = -3\hat{i} + 4\hat{j} - 5\hat{k}$ $\quad (c_1, c_2, c_3) = (-3, 4, -5)$

$\vec{b} - \vec{a} = \hat{i} - 2\hat{j} + 3\hat{k} - (2\hat{i} + 2\hat{j} + \hat{k})$

$= \hat{i} - 2\hat{j} + 3\hat{k} - 2\hat{i} - 2\hat{j} - 3\hat{k} = -\hat{i} - 4\hat{j} + 2\hat{k}$

Parametric Vector Equation: $\vec{r} = \vec{a} + s(\vec{b} - \vec{a}) + t\vec{c}$

$\vec{r} = (2\hat{i} + 2\hat{j} + \hat{k}) + s(-\hat{i} - 4\hat{j} + 2\hat{k}) + t(-3\hat{i} + 4\hat{j} - 5\hat{k})$

Cartesian Equation : $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$

$\begin{vmatrix} x - 2 & y - 2 & z - 1 \\ 1 - 2 & -2 - 2 & 3 - 1 \\ -3 & 4 & -5 \end{vmatrix} = 0$

$\Rightarrow \begin{vmatrix} x - 2 & y - 2 & z - 1 \\ -1 & -4 & 2 \\ -3 & 4 & -5 \end{vmatrix} = 0$

$\Rightarrow (x - 2)(20 - 8) - (y-2)(5 -+ 6) + (z - 1)(-4 - 12) = 0$

$\Rightarrow 12(x-2) - 11(y-2) - 16(z - 1) = 0$

$\Rightarrow 12x - 24 - 11y + 22 - 16z + 16 = 0$

$\Rightarrow 12x - 11y - 16z + 14 = 0 \Rightarrow 12x - 11y - 16z = -14$

Non Parametric Vector Equation:

$(x\hat{i} + y\hat{j} + z\hat{k}).(12\hat{i} - 11\hat{j} - 16\hat{k}) = -14$

$\Rightarrow \vec{r}.(12\hat{i} - 11\hat{j} - 16\hat{k}) = -14$

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5. Find the parametric form of vector equation, and Cartesian equations of the plane containing the line $\vec{r} = (\hat{i} - \hat{j} + 3\hat{k}) + t(2\hat{i} - \hat{j} + 4\hat{k})$ and perpendicular to plane $\vec{r} \cdot (\hat{i} + 2\hat{j} + \hat{k}) = 8$.

Solution:

containing the line $\vec{r} = (\hat{i} - \hat{j} + 3\hat{k}) + t(2\hat{i} - \hat{j} + 4\hat{k})$

Point : $\vec{a} = \hat{i} - \hat{j} + 3\hat{k}$ $\quad (x_1, y_1, z_1) = (1, -1, 3)$

Parallel Vector: $\vec{b} = 2\hat{i} - \hat{j} + 4\hat{k}$ $\quad (b_1, b_2, b_3) = (2, -1, 4)$

Parallel Vector: $\vec{c} = \hat{i} + 2\hat{j} + \hat{k}$ $\quad (c_1, c_2, c_3) = (1, 2, 1)$

Parametric Vector equation: $\vec{r} = \vec{a} + s\vec{b} + t\vec{c}$

$\Rightarrow \vec{r} = (\hat{i} - \hat{j} + 3\hat{k}) + s(2\hat{i} - \hat{j} + 4\hat{k}) + t(\hat{i} + 2\hat{j} + \hat{k})$

Cartesian Equation: $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$

$\begin{vmatrix} x - 1 & y + 1 & z - 3 \\ 2 & -1 & 4 \\ 1 & 2 & 1 \end{vmatrix} = 0$

$\Rightarrow (x - 1)(-1-8) - (y - 1)(2 - 4) + (z - 3)(4 + 1) = 0$

$\Rightarrow (x-1)(-9) - (y + 1)(-2) + (z - 3)(5) = 0$

$\Rightarrow -9(x-1) + 2(y + 1) + 5(z - 3) = 0$

$\Rightarrow -9x + 9 + 2y + 2 + 5z - 15 = 0$

$\Rightarrow -9x + 2y + 5z - 4 = 0 \Rightarrow 9x - 2y - 5z + 4 = 0$

Non Parametric Vector Equation:

$(x\hat{i} + y\hat{j} + z\hat{k}).(9\hat{i} - 2\hat{j} - 5\hat{k}) + 4 = 0$

$\Rightarrow \vec{r}.(9\hat{i} - 2\hat{j} - 5\hat{k}) + 4 = 0$

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6. Find the parametric vector , non-parametric vector and Cartesian form of the equations of the plane passing through the three non-collinear points (3,6,-2) , (-1,-2,6) , (6, 4,-2) .

Solution:

Point : $\vec{a} = 3\hat{i} + 6\hat{j} - 2\hat{k}$ $\quad (x_1, y_1, z_1) = (3, 6, -2)$

Point : $\vec{b} = -\hat{i} - 2\hat{j} + 6\hat{k}, (x_2, y_2, z_2) = (-1, -2, 6)$

Point : $\vec{c} = 6\hat{i} + 4\hat{j} - 2\hat{k}, (x_3, y_3, z_3) = (6, 4, -2)$

$\vec{b} - \vec{a} = -\hat{i} - 2\hat{j} + 6\hat{k} - 3\hat{i} - 6\hat{j} + 2\hat{k} = -4\hat{i} - 8\hat{j} + 8\hat{k}$

$\vec{c} - \vec{a} = 6\hat{i} + 4\hat{j} - 2\hat{k} - 3\hat{i} - 6\hat{j} + 2\hat{k} = 3\hat{i} - 2\hat{j} + 0\hat{k}$

Parametric Vector Equation:

$\vec{r} = \vec{a} + s(\vec{b} - \vec{a}) + t(\vec{c} - \vec{a})$

$\vec{r} = (3\hat{i} + 6\hat{j} - 2\hat{k}) + s(-4\hat{i} - 8\hat{j} + 8\hat{k}) + t(3\hat{i} - 2\hat{j} + 0\hat{k})$

Cartesian Equation : $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$

$\begin{vmatrix} x - 3 & y - 6 & z + 2 \\ -1 - 3 & -2 - 6 & 6 + 2 \\ 6 - 3 & 4 - 6 & -2 + 2 \end{vmatrix} = 0$

$\Rightarrow \begin{vmatrix} x - 3 & y - 6 & z + 2 \\ -4 & -8 & 8 \\ 3 & -2 & 0 \end{vmatrix} = 0$

$\Rightarrow (x - 3)(0 + 16) - (y - 6)(0 - 24) + (z + 2)(8 + 24) = 0$

$\Rightarrow 16(x-3) + 24(y-6) + 32(z + 2) = 0$

$\Rightarrow 16x - 48 + 24y - 144 + 32z + 64 = 0$

$\Rightarrow 16x + 24y + 32z - 128 = 0 \Rightarrow 2x + 3y + 4z - 16 = 0$

Non Parametric Vector Equation:

$(x\hat{i} + y\hat{j} + z\hat{k}).(2\hat{i} + 3\hat{j} + 4\hat{k}) = 16$

$\Rightarrow \vec{r}.(2\hat{i} + 3\hat{j} + 4\hat{k}) = 16$

Example 6.46

Show that the lines $\vec{r} = (-\hat{i} - 3\hat{j} - 5\hat{k}) + s(3\hat{i} + 5\hat{j} + 7\hat{k})$ and $\vec{r} = (2\hat{i} + 4\hat{j} + 6\hat{k}) + t(\hat{i} + 4\hat{j} + 7\hat{k})$ are coplanar. Also,find the non-parametric form of vector equation of the plane containing these lines.

Solution

$\vec{r} = \vec{a} + t\vec{b}$, $\quad \vec{r} = \vec{c} + s\vec{d}$

$\vec{a} = -\hat{i} - 3\hat{j} - 5\hat{k}$, $\quad \vec{b} = 3\hat{i} + 5\hat{j} + 7\hat{k}$,

$\vec{c} = 2\hat{i} + 4\hat{j} + 6\hat{k}$ and $\quad \vec{d} = \hat{i} + 4\hat{j} + 7\hat{k}$

We know that the two given lines are coplanar, if

$(\vec{c} - \vec{a}) \cdot (\vec{b} \times \vec{d}) = 0$

$$\vec{b} \times \vec{d} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 5 & 7 \\ 1 & 4 & 7 \end{vmatrix} = 7\hat{i} - 14\hat{j} + 7\hat{k}$$

$\vec{c} - \vec{a} = 2\hat{i} + 4\hat{j} + 6\hat{k} - (-\hat{i} - 3\hat{j} - 5\hat{k})$

$= 2\hat{i} + 4\hat{j} + 6\hat{k} + \hat{i} + 3\hat{j} + 5\hat{k} = 3\hat{i} + 7\hat{j} + 11\hat{k}$

$(\vec{c} - \vec{a}) \cdot (\vec{b} \times \vec{d}) = (3\hat{i} + 7\hat{j} + 11\hat{k}) \cdot (7\hat{i} - 14\hat{j} + 7\hat{k})$

$= 21-98+77 = 98-98 = 0$

$(\vec{r} - \vec{a}) \cdot (\vec{b} \times \vec{d}) = 0$

$(\vec{r} - (-\hat{i} - 3\hat{j} - 5\hat{k})) \cdot (7\hat{i} - 14\hat{j} + 7\hat{k}) = 0.$

$\vec{r} \cdot (7\hat{i} - 14\hat{j} + 7\hat{k}) - (-\hat{i} - 3\hat{j} - 5\hat{k})) \cdot (7\hat{i} - 14\hat{j} + 7\hat{k}) = 0$

$\vec{r} \cdot (7\hat{i} - 14\hat{j} + 7\hat{k}) - (-7 + 42 - 35) = 0$

$\vec{r} \cdot (7\hat{i} - 14\hat{j} + 7\hat{k}) = 0 \quad (\div \text{ by } 7) \quad \vec{r} \cdot (\hat{i} - 2\hat{j} + \hat{k}) = 0$

EX 6.8 (1).

Show that the straight lines $\vec{r} = (5\hat{i} + 7\hat{j} - 3\hat{k}) + s(4\hat{i} + 4\hat{j} - 5\hat{k})$ and $\vec{r} = (8\hat{i} + 4\hat{j} + 5\hat{k}) + t(7\hat{i} + \hat{j} + 3\hat{k})$ are coplanar. Find the vector equation of the plane in which they lie.

Solution:

Let $\vec{a} = 5\vec{i} + 7\vec{j} - 3\vec{k}$ $\quad \vec{b} = 4\vec{i} + 4\vec{j} - 5\vec{k}$

$\vec{c} = 8\vec{i} + 4\vec{j} + 5\vec{k}$ $\quad \vec{d} = 7\vec{i} + \vec{j} + 3\vec{k}$

For coplanar $(\vec{c} - \vec{a}) \cdot (\vec{b} \times \vec{d}) = 0$

$$\vec{b} \times \vec{d} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 4 & -5 \\ 7 & 1 & 3 \end{vmatrix} = \vec{i}(12 + 5) - \vec{j}(12 + 35) + \vec{k}(4 - 28)$$

$\vec{b} \times \vec{d} = 17\vec{i} - 47\vec{j} - 24\vec{k}$

$\vec{c} - \vec{a} = (8\vec{i} + 4\vec{j} + 5\vec{k}) - (5\vec{i} + 7\vec{j} - 3\vec{k}) = 3\vec{i} - 3\vec{j} + 8\vec{k}$

$(1) \Rightarrow (3\vec{i} - 3\vec{j} + 8\vec{k}) \cdot (17\vec{i} - 47\vec{j} - 24\vec{k}) = 51 + 141 - 192 = 0$

$\therefore$ The two given lines are coplanar so,

the non-parametric vector equation is $(\vec{r} - \vec{a}) \cdot (\vec{b} \times \vec{d}) = 0$

$\vec{r} \cdot (\vec{b} \times \vec{d}) = \vec{a} \cdot (\vec{b} \times \vec{d})$

$\vec{r} \cdot (17\vec{i} - 47\vec{j} - 24\vec{k}) = (5\vec{i} + 7\vec{j} - 3\vec{k}).(17\vec{i} - 47\vec{j} - 24\vec{k})$

$\vec{r} \cdot (17\vec{i} - 47\vec{j} - 24\vec{k}) = 85 - 329 + 72$

$\Rightarrow \vec{r} \cdot (17\vec{i} - 47\vec{j} - 24\vec{k}) = -172$

EX 6.8 (2).

Show that lines $\frac{x-2}{1} = \frac{y-3}{1} = \frac{z-4}{3}$ and $\frac{x-1}{-3} = \frac{y-4}{2} = \frac{z-5}{1}$ are coplanar. Also, find the plane containing these lines.

Solution: From the lines we have,

$(x_1, y_1, z_1) = (2,3,4)$ & $(x_2, y_2, z_2) = (1,4,5)$

$(b_1, b_2, b_3) = (1,1,3)$ &$(d_1, d_2, d_3) = (-3,2,1)$

Condition for coplanarity: $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$

$$= \begin{vmatrix} -1 & 1 & 1 \\ 1 & 1 & 3 \\ -3 & 2 & 1 \end{vmatrix} = -(1 - 6) - 1(1 + 9) + 1(2 + 3)$$

$= 5 - 10 + 5 = 0$

$\therefore$ The given two lines are coplanar.

Cartesian form of equation of the plane containing the two given coplanar lines.

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} x - 2 & y - 3 & z - 4 \\ 1 & 1 & 3 \\ -3 & 2 & 1 \end{vmatrix} = 0$$

$(x - 2)[1 - 6] - (y - 3)[1 + 9] + (z - 4)[2 + 3] = 0$

$-5(x - 2) - 10(y - 3) + 5(z - 4) = 0$

$-5x + 10 - 10y + 30 + 5z - 20 = 0$

$-5x - 10y + 5z + 20 = 0$

$(\div \text{ by } -5) \Rightarrow x + 2y - z - 4 = 0$

EX 6.8 (4).

If the straight lines $\frac{x-1}{2} = \frac{y+1}{\lambda} = \frac{z}{2}$ and $\frac{x+1}{5} = \frac{y+1}{2} = \frac{z}{\lambda}$: are coplanar, find λ and equation of the planes containing these two lines.

Solution: From the lines we have,

$(x_1, y_1, z_1) = (1, -1,0)$ and $(x_2, y_2, z_2) = (-1, -1,0)$

$(b_1, b_2, b_3) = (2, \lambda, 2)$ and $(d_1, d_2, d_3) = (5,2,\lambda)$

Condition for coplanarity

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$$

$$\begin{vmatrix} -2 & 0 & 0 \\ 2 & \lambda & 2 \\ 5 & 2 & \lambda \end{vmatrix} = 0 \Rightarrow -2(\lambda^2 - 4) = 0 \text{ , } \lambda^2 = 4 \Rightarrow \lambda = \pm 2$$

(i) If $\lambda=2$ $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$

$$\begin{vmatrix} x - 1 & y + 1 & z \\ 2 & 2 & 2 \\ 5 & 2 & 2 \end{vmatrix} = 0$$

$(x - 1)[0] - (y + 1)[4 - 10] + z[4 - 10] = 0$

$6(y + 1) - 6(z) = 0 \Rightarrow 6y + 6 - 6z = 0$

$(\div \text{ by } 6) \Rightarrow (y - z + 1) = 0$

(ii) If $\lambda=-2$ $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$

$$\begin{vmatrix} x - 1 & y + 1 & z \\ 2 & -2 & 2 \\ 5 & 2 & -2 \end{vmatrix} = 0$$

$(x - 1)[0] - (y + 1)[-4 - 10] + z[4 + 10] = 0$

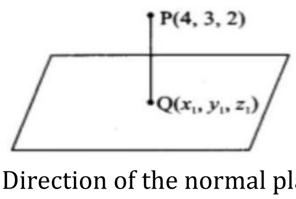
$14(y + 1) + 14z = 0 \Rightarrow 14y + 14 + 14z = 0$

$(\div \text{ by } 14) \Rightarrow y + z + 1 = 0$

EX 6.9 (8).

Find the coordinates of the foot of the perpendicular and length of the perpendicular from the point $(4, 3, 2)$ to the plane $x + 2y + 3z = 2$

Solution:



Direction of the normal plane (1,2,3)

d.c.s of the PQ is (1,2,3) $\therefore$Eqn of PQ; $\frac{x_1-4}{1} = \frac{y_1-3}{2} = \frac{z_1-2}{3} = k$

$x_1 = k + 4, y_1 = 2k + 3, z_1 = 3k + 2$

This passes through the plane $x + 2y + 3z = 2$

$k + 4 + 2(2k + 3) + 3(3k + 2) = 2$

$k + 4 + 4k + 6 + 9k + 6 = 2$

$14k = 2 - 16 \Rightarrow 14k = -14 \Rightarrow k = -1$

$\therefore$ The coordinate of the foot of the perpendicular is $(3,1, -1)$

$\therefore$ Length of the perpendicular to the plane is

$$= \left| \frac{4+2(3)+3(2)-2}{\sqrt{(1)^2+(2)^2+(3)^2}} \right| = \left| \frac{4+6+6-2}{\sqrt{1+4+9}} \right| = \frac{14}{\sqrt{14}} = \sqrt{14} \text{ units}$$