SCHOOL EDUCTION DEPARTMENT

CHENNAI   DISTRICT

LEARNING MATERIAL
2023-2024

HIGHER SECONDARY SECOND YEAR

MATHEMATICS

Preface

We convey our sincere gratitude to our respected Chief Educational Officer, who has given this opportunity to bring out an unique material for the students (XII standard Maths) in the name of Learning Material.

The minimum learning material is prepared based on the selected chapters. This includes classification for selected chapters,solved textbook exercise problems(2 marks, 3 marks and 5 marks).

Students can prepare the example problems based on the classification. All the text book MCQ problems have to be practiced regularly. Students must practice all the problems in the classification. This material mainly focus on the slow learners to achieve their goals.

Good effort always lead to success

All the best!!!

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CHAPTER 1: MATRICES AND DETERMINANTS

2 - MARKS, 3 -MARKS (5- MARKS ONLY QUESTIONS GIVEN)

2 MARKS

EXERCISE 1.1 : 1(i). Find the adjoint of $\begin{pmatrix} -3 & 4 \\ 6 & 2 \end{pmatrix}$

Solution:

$A = \begin{pmatrix} -3 & 4 \\ 6 & 2 \end{pmatrix}$

adj A = $(A_c)^T = \begin{pmatrix} 2 & -4 \\ -6 & -3 \end{pmatrix}$

Exercise 1.1 (2) (i) : Find the inverse of $\begin{bmatrix} -2 & 4 \\ 1 & -3 \end{bmatrix}$

Solution: $A = \begin{bmatrix} -2 & 4 \\ 1 & -3 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|}(\text{Adj } A)$

$|A| = \begin{vmatrix} -2 & 4 \\ 1 & -3 \end{vmatrix} = 6 - 4 = 2$

Adj A $= \begin{bmatrix} -3 & -4 \\ -1 & -2 \end{bmatrix}$

$A^{-1} = \frac{1}{2}\begin{bmatrix} -3 & -4 \\ -1 & -2 \end{bmatrix}$ $(\textit{Using formula})$

Exercise 1.1(9): If adj A = $\begin{pmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{pmatrix}$, find A^{-1}

Solution:

adj A = $\begin{pmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{pmatrix}$

|adj A| = $\begin{vmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{vmatrix}$ = 0 + 2(36-18) + 0 = 2(18) = 36

$A^{-1} = \pm\frac{1}{\sqrt{|\text{adj } A|}}(\text{adj } A) = \pm\frac{1}{\sqrt{36}}\begin{pmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{pmatrix}$

$= \pm\frac{1}{6}\begin{pmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{pmatrix}$

Exercise 1.2 (1)(i):

Find the rank of the matrix by minor method: $\begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$

Solution:

$A = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$; A is a mattrix of order 2X2; ρ(A) ≤ min{2,2} = 2

$|A| = \begin{vmatrix} 2 & -4 \\ -1 & 2 \end{vmatrix}$ = 4 – 4 = 0 ; ρ(A) ≠ 2 ⟹ ρ(A) < 2

a_{11} = 2 ≠ 0 ⟹ Since 1 x 1 minor not equal to zero ρ(A) = 1

Exercise 1.2 (1)(ii):

Find the rank of the matrix by minor method: $\begin{pmatrix} -1 & 3 \\ 4 & -7 \\ 3 & -4 \end{pmatrix}$

Solution:

$A = \begin{pmatrix} -1 & 3 \\ 4 & -7 \\ 3 & -4 \end{pmatrix}$

A is a mattrix of order 3 X 2; ρ(A) ≤ min{3,2} = 2

$|A| = \begin{vmatrix} -1 & 3 \\ 4 & -7 \end{vmatrix}$ = 7 – 12 = -5 ≠ 0 ;

Since 2 x 2 minor not equal to zero ⟹ ρ(A) = 2

Exercise 1.2 (1)(iii):

Find the rank of matrix by minor method: $\begin{pmatrix} 1 & -2 & -1 & 0 \\ 3 & -6 & -3 & 1 \end{pmatrix}$

Solution:

$A = \begin{pmatrix} 1 & -2 & -1 & 0 \\ 3 & -6 & -3 & 1 \end{pmatrix}$

A is a mattrix of order 2 X 4; ρ(A) ≤ min{4,2} = 2

Consider $\Delta_1 = \begin{vmatrix} 1 & -2 \\ 3 & -6 \end{vmatrix}$ = -6 + 6 = 0

Consider $\Delta_2 = \begin{vmatrix} -2 & -1 \\ -6 & -3 \end{vmatrix}$ = 6 – 6 = 0

We must find all possible 2 x 2 minors of A check |A| ≠ 0

Consider $\Delta_3 = \begin{vmatrix} -1 & 0 \\ -3 & 1 \end{vmatrix}$ = -1 – 0 = -1 ≠ 0

Since 2 x 2 minor not equal to zero ⟹ ρ(A) = 2

Exercise 1.2 (1)(iv):

Find the rank of the matrix by minor method:

$\begin{pmatrix} 1 & -2 & 3 \\ 2 & 4 & -6 \\ 5 & 1 & -1 \end{pmatrix}$

Solution:

$A = \begin{pmatrix} 1 & -2 & 3 \\ 2 & 4 & -6 \\ 5 & 1 & -1 \end{pmatrix}$

A is a mattrix of order 3 X 3; ρ(A) ≤ min {3,3} = 3

$\Delta_1 = \begin{vmatrix} 1 & -2 & 3 \\ 2 & 4 & -6 \\ 5 & 1 & -1 \end{vmatrix}$

= 1(-4+6) + 2(-2+30) + 3(2 – 20)

= 1(2) + 2(28) + 3(-18) = 2 + 56 – 54 = 4 ≠ 0

Since 3 x 3 minor not equal to zero ⟹ ρ(A) = 3

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3 MARK QUESTIONS

EXERCISE 1.1

1(ii) Find the adjoint of $\begin{pmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{pmatrix}$ –

$$\begin{array}{cccccc} 2 & 3 & 1 & 2 & 3 & 1 \\ 3 & 4 & 1 & 3 & 4 & 1 \\ 3 & 7 & 2 & 3 & 7 & 2 \\ 2 & 3 & 1 & 2 & 3 & 1 \\ 3 & 4 & 1 & 3 & 4 & 1 \\ 3 & 7 & 2 & 3 & 7 & 2 \end{array}$$

Solution: $\mathbf{A} = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{pmatrix}$

$$\text{adj A} = (\text{A}_c)^T = \begin{bmatrix} 8-7 & 3-6 & 21-12 \\ 7-6 & 4-3 & 9-14 \\ 3-4 & 3-2 & 8-9 \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & -3 & 9 \\ 1 & 1 & -5 \\ -1 & 1 & -1 \end{bmatrix}^T = \begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix}$$

1(iii).

Find the adjoint of $\frac{1}{3}\begin{pmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{pmatrix}$

$$\begin{array}{cccccc} 2 & 2 & 1 & 2 & 2 & 1 \\ -2 & 1 & 2 & -2 & 1 & 2 \\ 1 & -2 & 2 & 1 & -2 & 2 \\ 2 & 2 & 1 & 2 & 2 & 1 \\ -2 & 1 & 2 & -2 & 1 & 2 \\ 1 & -2 & 2 & 1 & -2 & 2 \end{array}$$

Solution: $\text{A} = \frac{1}{3}\begin{pmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{pmatrix}$

$$\text{Adj A} = \left(\tfrac{1}{3}\right)^2 \begin{bmatrix} 2+4 & 2+4 & 4-1 \\ -2-4 & 4-1 & 2+4 \\ 4-1 & -4-2 & 2+4 \end{bmatrix}^T$$

$$= \frac{1}{9}\begin{bmatrix} 6 & 6 & 3 \\ -6 & 3 & 6 \\ 3 & -6 & 6 \end{bmatrix}^T = \frac{1}{9}\begin{bmatrix} 6 & -6 & 3 \\ 6 & 3 & -6 \\ 3 & 6 & 6 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{bmatrix}$$

Exercise 1.1 (2) (iii): Find the inverse of $\begin{pmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{pmatrix}$

Solution: Let $\text{A} = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{pmatrix}$

$|A| = 2(8 - 7) - 3(6 - 3) + 1(21 - 12) = 2(1) - 3(3) + 1(9)$

$= 2 - 9 + 9 = 2$ (inverse exists)

$$\text{adj A} = (\text{A}_c)^T = \begin{bmatrix} 8-7 & 3-6 & 21-12 \\ 7-6 & 4-3 & 9-14 \\ 3-4 & 3-2 & 8-9 \end{bmatrix}^T$$

$$\begin{array}{cccccc} 2 & 3 & 1 & 2 & 3 & 1 \\ 3 & 4 & 1 & 3 & 4 & 1 \\ 3 & 7 & 2 & 3 & 7 & 2 \\ 2 & 3 & 1 & 2 & 3 & 1 \\ 3 & 4 & 1 & 3 & 4 & 1 \\ 3 & 7 & 2 & 3 & 7 & 2 \end{array}$$

$$= \begin{bmatrix} 1 & -3 & 9 \\ 1 & 1 & -5 \\ -1 & 1 & -1 \end{bmatrix}^T = \begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}(\text{Adj A}) = \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix}$$

Practise Ex 1.1 2(ii)

Exercise 1.1 (5): If $\text{A} = \frac{1}{9}\begin{pmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{pmatrix}$, prove that $A^{-1} = A^T$

Solution:

$\text{A} = \frac{1}{9}\begin{pmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{pmatrix}$ and $\text{A}^T = \frac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix}$

$$\text{A A}^T = \frac{1}{9}\begin{pmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{pmatrix} \frac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix}$$

$$= \frac{1}{81}\begin{bmatrix} 64+1+16 & -32+4+28 & -8-8+16 \\ -32+4+28 & 16+16+49 & 4-32+28 \\ -8-8+16 & 4-32+28 & 1+64+16 \end{bmatrix}$$

$$= \frac{1}{81}\begin{bmatrix} 81 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 81 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} = I_3$$

$$\text{A}^T\text{A} = \frac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix} \frac{1}{9}\begin{pmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{pmatrix}$$

$$= \frac{1}{81}\begin{bmatrix} 64+16+1 & -8+16-8 & -32+28+4 \\ -8+16-8 & 1+16+64 & 4+28-32 \\ -32+28+4 & 4+28-32 & 16+49+16 \end{bmatrix}$$

$$= \frac{1}{81}\begin{bmatrix} 81 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 81 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} = I_3$$

$\text{A A}^T = \text{A}^T\text{A} = I_3 \implies A^{-1} = A^T$

Exercise 1.1(6):

If $\text{A} = \begin{pmatrix} 8 & -4 \\ -5 & 3 \end{pmatrix}$, verify that $\text{A(adj A)} = \text{(adj A)A} = |A|I_2$

Solution:

$\text{A} = \begin{pmatrix} 8 & -4 \\ -5 & 3 \end{pmatrix}$ *and* $\text{adj A} = \begin{pmatrix} 3 & 4 \\ 5 & 8 \end{pmatrix}$

$$\text{A(adj A)} = \begin{pmatrix} 8 & -4 \\ -5 & 3 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ 5 & 8 \end{pmatrix} = \begin{pmatrix} 24-20 & 32-32 \\ -15+15 & -20+24 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$$

$$\text{(adj A)A} = \begin{pmatrix} 3 & 4 \\ 5 & 8 \end{pmatrix}\begin{pmatrix} 8 & -4 \\ -5 & 3 \end{pmatrix} = \begin{pmatrix} 24-20 & -12+12 \\ 40-40 & -20+24 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$$

$|A| = \begin{vmatrix} 8 & -4 \\ -5 & 3 \end{vmatrix} = 24 - 20 = 4$

$|A|I_2 = 4\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$

$\text{A(adj A)} = \text{(adj A)A} = |A|I_2 = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$. Hence proved

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Exercise 1.1(9): If $\text{adj A} = \begin{pmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{pmatrix}$, find A^{-1}

Solution:

$$\text{adj A} = \begin{pmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{pmatrix}$$

$$|\text{adj A}| = \begin{vmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{vmatrix} = 0 + 2(36\text{-}18) + 0 = 2(18) = 36$$

$$A^{-1} = \pm\frac{1}{\sqrt{|\text{ adj A}|}}(\text{adj A})$$

$$= \pm\frac{1}{\sqrt{36}}\begin{pmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{pmatrix}$$

$$= \pm\frac{1}{6}\begin{pmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{pmatrix}$$

Exercise 1.1 (7): $A = \begin{pmatrix}3 & 2\\ 7 & 5\end{pmatrix}$ and $B = \begin{pmatrix}-1 & -3\\ 5 & 2\end{pmatrix}$.

verify that $(AB)^{-1} = B^{-1}A^{-1}$

Solution: $A = \begin{pmatrix}3 & 2\\ 7 & 5\end{pmatrix}$, $B = \begin{pmatrix}-1 & -3\\ 5 & 2\end{pmatrix}$

$$AB = \begin{pmatrix}3 & 2\\ 7 & 5\end{pmatrix}\begin{pmatrix}-1 & -3\\ 5 & 2\end{pmatrix} = \begin{pmatrix}-3+10 & -9+4\\ -7+25 & -21+10\end{pmatrix}$$

$$= \begin{pmatrix}7 & -5\\ 18 & -11\end{pmatrix}$$

$$|AB| = \begin{vmatrix}7 & -5\\ 18 & -11\end{vmatrix} = -77 + 90 = 13$$

$$\text{adj}(AB) = \begin{pmatrix}-11 & 5\\ -18 & 7\end{pmatrix}$$

$$(AB)^{-1} = \frac{1}{|AB|}(\text{Adj } AB) = \frac{1}{13}\begin{pmatrix}-11 & 5\\ -18 & 7\end{pmatrix}$$

$$B = \begin{pmatrix}-1 & -3\\ 5 & 2\end{pmatrix} \text{ and } |B| = \begin{vmatrix}-1 & -3\\ 5 & 2\end{vmatrix} = -2 + 15 = 13$$

$$\text{adj } B = \begin{pmatrix}2 & 3\\ -5 & -1\end{pmatrix}$$

$$B^{-1} = \frac{1}{|B|}(\text{Adj } B) = \frac{1}{13}\begin{pmatrix}2 & 3\\ -5 & -1\end{pmatrix}$$

$$A = \begin{pmatrix}3 & 2\\ 7 & 5\end{pmatrix} \text{ and } |A| = \begin{vmatrix}3 & 2\\ 7 & 5\end{vmatrix} = 15 - 14 = 1$$

$$\text{adj } A = \begin{pmatrix}5 & -2\\ -7 & 3\end{pmatrix}$$

$$A^{-1} = \frac{1}{|A|}(\text{Adj } A) = \frac{1}{1}\begin{pmatrix}5 & -2\\ -7 & 3\end{pmatrix}$$

$$B^{-1}A^{-1} = \frac{1}{13}\begin{pmatrix}2 & 3\\ -5 & -1\end{pmatrix}\frac{1}{1}\begin{pmatrix}5 & -2\\ -7 & 3\end{pmatrix} = \frac{1}{13}\begin{pmatrix}2 & 3\\ -5 & -1\end{pmatrix}\begin{pmatrix}5 & -2\\ -7 & 3\end{pmatrix}$$

$$= \frac{1}{13}\begin{pmatrix}10-21 & -4+9\\ -25+7 & 10-3\end{pmatrix} = \frac{1}{13}\begin{pmatrix}-11 & 5\\ -18 & 7\end{pmatrix}$$

$(AB)^{-1} = B^{-1}A^{-1} = \frac{1}{13}\begin{pmatrix}-11 & 5\\ -18 & 7\end{pmatrix}$ Hence proved.

Exercise 1.1(8): If $\text{adj}A = \begin{pmatrix}2 & -4 & 2\\ -3 & 12 & -7\\ -2 & 0 & 2\end{pmatrix}$, find A

Solution: $A = \pm\frac{1}{\sqrt{|\text{adj } A|}}\text{adj}(\text{adj } A)$

$$|\text{Adj } A| = \begin{vmatrix}2 & -4 & 2\\ -3 & 12 & -7\\ -2 & 0 & 2\end{vmatrix}$$

$$\begin{array}{cccccc} 2 & -4 & 2 & 2 & -4 & 2\\ -3 & 12 & -7 & -3 & 12 & -7\\ -2 & 0 & 2 & -2 & 0 & 2\\ 2 & -4 & 2 & 2 & -4 & 2\\ -3 & 12 & -7 & -3 & 12 & -7\\ -2 & 0 & 2 & -2 & 0 & 2 \end{array}$$

$= 2(24-0)-(-4)(-6-14)+2(0+24)$

$= 2(24) + 4(-20) + 2(24)$

$= 48 - 80 + 48 = 96 - 80 = 16$

$$\text{adj}(\text{adj } A) = \begin{bmatrix}24-0 & 14+6 & 0+24\\ 0+8 & 4+4 & 8-0\\ 28-24 & -6+14 & 24-12\end{bmatrix}^T$$

$$= \begin{bmatrix}24 & 20 & 24\\ 8 & 8 & 8\\ 4 & 8 & 12\end{bmatrix}^T = \begin{bmatrix}24 & 8 & 4\\ 20 & 8 & 8\\ 24 & 8 & 12\end{bmatrix}$$

$$A = \pm\frac{1}{\sqrt{16}}\begin{bmatrix}24 & 8 & 4\\ 20 & 8 & 8\\ 24 & 8 & 12\end{bmatrix}$$

$$= \pm\frac{1}{4}4\begin{bmatrix}6 & 2 & 1\\ 5 & 2 & 2\\ 6 & 2 & 3\end{bmatrix}$$

$$= \pm\begin{bmatrix}6 & 2 & 1\\ 5 & 2 & 2\\ 6 & 2 & 3\end{bmatrix}$$

Exercise 1.1(10): Find adj(adj A), if $\text{adj } A = \begin{pmatrix}1 & 0 & 1\\ 0 & 2 & 0\\ -1 & 0 & 1\end{pmatrix}$

SOLUTION:

$$\text{adj } A = \begin{pmatrix}1 & 0 & 1\\ 0 & 2 & 0\\ -1 & 0 & 1\end{pmatrix}$$

$$\begin{array}{cccccc} 1 & 0 & 1 & 1 & 0 & 1\\ 0 & 2 & 0 & 0 & 2 & 0\\ -1 & 0 & 1 & -1 & 0 & 1\\ 1 & 0 & 1 & 1 & 0 & 1\\ 0 & 2 & 0 & 0 & 2 & 0\\ -1 & 0 & 1 & -1 & 0 & 1 \end{array}$$

$$\text{Adj}(\text{adj } A) = \begin{bmatrix}2-0 & 0-0 & 0+2\\ 0-0 & 1+1 & 0-0\\ 0-2 & 0-0 & 2-0\end{bmatrix}^T$$

$$= \begin{bmatrix}2 & 0 & 2\\ 0 & 2 & 0\\ -2 & 0 & 2\end{bmatrix}^T = \begin{bmatrix}2 & 0 & -2\\ 0 & 2 & 0\\ 2 & 0 & 2\end{bmatrix}$$

Exercise 1.1(11): $A = \begin{pmatrix}1 & \tan x\\ -\tan x & 1\end{pmatrix}$, show that $A^TA^{-1} = \begin{pmatrix}\cos 2x & -\sin 2x\\ \sin 2x & \cos 2x\end{pmatrix}$

SOLUTION: $|A| = 1 + \tan^2 x = \sec^2 x$

$$\text{Adj } A = \begin{pmatrix}1 & -\tan x\\ \tan x & 1\end{pmatrix} \text{ and } A^T = \begin{pmatrix}1 & -\tan x\\ \tan x & 1\end{pmatrix}$$

$$A^{-1} = \frac{1}{|A|}(\text{Adj } A)$$

$$A^{-1} = \frac{1}{\sec^2 x|}\begin{pmatrix}1 & -\tan x\\ \tan x & 1\end{pmatrix}$$

$$A^TA^{-1} = \frac{1}{\sec^2 x}\begin{pmatrix}1 & -\tan x\\ \tan x & 1\end{pmatrix}\begin{pmatrix}1 & -\tan x\\ \tan x & 1\end{pmatrix}$$

$$= \cos^2 x\begin{pmatrix}1 - \tan^2 x & -\tan x - \tan x\\ \tan x + \tan x & -\tan^2 x + 1\end{pmatrix}$$

$$= \cos^2 x\begin{pmatrix}1 - \frac{\sin^2 x}{\cos^2 x} & -2\tan x\\ 2\tan x & 1 - \frac{\sin^2 x}{\cos^2 x}\end{pmatrix}$$

$$= \begin{pmatrix}\cos^2 x - \frac{\sin^2 x\cos^2 x}{\cos^2 x} & -2\tan x\cos^2 x\\ 2\tan x\cos^2 x & \cos^2 x - \frac{\sin^2 x\cos^2 x}{\cos^2 x}\end{pmatrix}$$

$$= \begin{pmatrix}\cos^2 x - \sin^2 x & -2\sin x\cos x\\ 2\sin x\cos x & \cos^2 x - \sin^2 x\end{pmatrix}$$

$$= \begin{pmatrix}\cos 2x & -\sin 2x\\ \sin 2x & \cos 2x\end{pmatrix}$$

Exercise 1.1(12):

Find the matrix A for which $A\begin{pmatrix}5 & 3\\ -1 & -2\end{pmatrix} = \begin{pmatrix}14 & 7\\ 7 & 7\end{pmatrix}$

SOLUTION:

$$A\begin{pmatrix}5 & 3\\ -1 & -2\end{pmatrix} = \begin{pmatrix}14 & 7\\ 7 & 7\end{pmatrix}$$

$A \quad B \quad = C \Rightarrow A = C B^{-1}$

$$B = \begin{pmatrix}5 & 3\\ -1 & -2\end{pmatrix} \Rightarrow |B| = -10 + 3 = -7$$

$$\text{Adj } B = \begin{pmatrix}-2 & -3\\ 1 & 5\end{pmatrix} \Rightarrow B^{-1} = \frac{1}{-7}\begin{pmatrix}-2 & -3\\ 1 & 5\end{pmatrix}$$

$$A = C B^{-1} = \frac{1}{-7}\begin{pmatrix}14 & 7\\ 7 & 7\end{pmatrix}\begin{pmatrix}-2 & -3\\ 1 & 5\end{pmatrix}$$

$$= \frac{1}{-7}\begin{pmatrix}-28+7 & -42+35\\ -14+7 & -21+35\end{pmatrix}$$

$$= \frac{1}{-7}\begin{pmatrix}-21 & -7\\ -7 & 14\end{pmatrix} = \begin{pmatrix}\frac{-21}{-7} & \frac{-7}{-7}\\ \frac{-7}{-7} & \frac{14}{-7}\end{pmatrix} = \begin{pmatrix}3 & 1\\ 1 & -2\end{pmatrix}$$

Exercise 1.1(13):

Given $A = \begin{pmatrix}1 & -1\\ 2 & 0\end{pmatrix}$, $B=\begin{pmatrix}3 & -2\\ 1 & 1\end{pmatrix}$, and $C = \begin{pmatrix}1 & 1\\ 2 & 2\end{pmatrix}$,

Find a matrix X such that AXB = C.

Solution:

$AXB = C \Rightarrow A^{-1}\,(AXB)B^{-1} = A^{-1}C\,B^{-1}$

$\Rightarrow (A^{-1}\,A)X(BB^{-1}) = A^{-1}C\,B^{-1}$

$\Rightarrow X = A^{-1}C\,B^{-1}$

$A = \begin{pmatrix}1 & -1\\ 2 & 0\end{pmatrix} \Rightarrow |A| = 0+2 = 2$ & $\text{Adj } A = \begin{pmatrix}0 & 1\\ -2 & 1\end{pmatrix}$

$A^{-1} = \frac{1}{|A|}(\text{Adj } A) = \frac{1}{2}\begin{pmatrix}0 & 1\\ -2 & 1\end{pmatrix}$

$B=\begin{pmatrix}3 & -2\\ 1 & 1\end{pmatrix} \Rightarrow |B| = 3+2 = 5$ & $\text{Adj } B = \begin{pmatrix}1 & 2\\ -1 & 3\end{pmatrix}$

$B^{-1} = \frac{1}{|B|}(\text{Adj } B) = \frac{1}{5}\begin{pmatrix}1 & 2\\ -1 & 3\end{pmatrix}$

$X = \frac{1}{2}\begin{pmatrix}0 & 1\\ -2 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 2 & 2\end{pmatrix}\frac{1}{5}\begin{pmatrix}1 & 2\\ -1 & 3\end{pmatrix}$

$= \frac{1}{10}\begin{pmatrix}0 & 1\\ -2 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 2 & 2\end{pmatrix}\begin{pmatrix}1 & 2\\ -1 & 3\end{pmatrix}$

$= \frac{1}{10}\begin{pmatrix}0+2 & 0+2\\ -2+2 & -2+2\end{pmatrix}\begin{pmatrix}1 & 2\\ -1 & 3\end{pmatrix} = \frac{1}{10}\begin{pmatrix}2 & 2\\ 0 & 0\end{pmatrix}\begin{pmatrix}1 & 2\\ -1 & 3\end{pmatrix}$

$= \frac{1}{10}\begin{pmatrix}2-2 & 4+6\\ 0+0 & 0+0\end{pmatrix} = \frac{1}{10}\begin{pmatrix}0 & 10\\ 0 & 0\end{pmatrix} = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$

Exercise 1.1 (14): If $A = \begin{pmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{pmatrix}$, show that $A^{-1} = \frac{1}{2}(A^2 - 3I)$

Solution:

$A = \begin{pmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{pmatrix}$

0	1	1	0	1	1
1	0	1	1	0	1
1	1	0	1	1	0
0	1	1	0	1	1
1	0	1	1	0	1
1	1	0	1	1	0

$\begin{vmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{vmatrix}$ $= 0\,(0\text{-}1) - 1\,(0\text{-}1) + 1(1\text{-}0)$

$= 0(-1) - 1(-1) + 1(1)$

$= 0 + 1 + 1 = 2$

$\text{Adj } A = (A_c)^T = \begin{bmatrix}0-1 & 1-0 & 1-0\\ 1-0 & 0-1 & 1-0\\ 0-1 & 1-0 & 0-1\end{bmatrix}^T = \begin{bmatrix}-1 & 1 & 1\\ 1 & -1 & 1\\ -1 & 1 & -1\end{bmatrix}^T$

$= \begin{bmatrix}-1 & 1 & -1\\ 1 & -1 & 1\\ 1 & 1 & -1\end{bmatrix}$

$A^{-1} = \frac{1}{|A|}(\text{Adj } A) = \frac{1}{2}\begin{bmatrix}-1 & 1 & -1\\ 1 & -1 & 1\\ 1 & 1 & -1\end{bmatrix}$

$A^2 = A \times A = \begin{pmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{pmatrix}$

$= \begin{bmatrix}0+1+1 & 0+0+1 & 0+1+0\\ 0+0+1 & 1+0+1 & 1+0+0\\ 0+1+0 & 1+0+0 & 1+1+0\end{bmatrix}$

$= \begin{pmatrix}2 & 1 & 1\\ 1 & 2 & 1\\ 1 & 1 & 2\end{pmatrix}$

$A^2 - 3I = \begin{pmatrix}2 & 1 & 1\\ 1 & 2 & 1\\ 1 & 1 & 2\end{pmatrix} - 3\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$

$= \begin{pmatrix}2 & 1 & 1\\ 1 & 2 & 1\\ 1 & 1 & 2\end{pmatrix} + \begin{pmatrix}-3 & 0 & 0\\ 0 & -3 & 0\\ 0 & 0 & -3\end{pmatrix}$

$= \begin{pmatrix}-1 & 1 & 1\\ 1 & -1 & 1\\ 1 & 1 & -1\end{pmatrix}$

$\frac{1}{2}(A^2 - 3I) = \frac{1}{2}\begin{pmatrix}-1 & 1 & 1\\ 1 & -1 & 1\\ 1 & 1 & -1\end{pmatrix} = A^{-1}$

Exercise 1.2 (2) (i):

Find the rank of the matrix by row reduction method:

$\begin{pmatrix}1 & 1 & 1 & 3\\ 2 & -1 & 3 & 4\\ 5 & -1 & 7 & 11\end{pmatrix}$

Solution:

$A = \begin{bmatrix}1 & 1 & 1 & 3\\ 2 & -1 & 3 & 4\\ 5 & -1 & 7 & 11\end{bmatrix}$

$\sim \begin{bmatrix}1 & 1 & 1 & 3\\ 0 & -3 & 1 & -2\\ 0 & -6 & 2 & -4\end{bmatrix}$ $R_2 \to R_2 - 2R_1;\ R_3 \to R_3 - 5R_1$

$\sim \begin{bmatrix}1 & 1 & 1 & 3\\ 0 & -3 & 1 & -2\\ 0 & 0 & 0 & 0\end{bmatrix}$ $R_3 \to R_3 - 2R_2$

This is in echelon form; no of nonzero rows = 2 $\Longrightarrow \rho(A) = 2$

Exercise 1.2 (2) (ii):

Find the rank of the matrices by row reduction method:

$\begin{pmatrix}1 & 2 & -1\\ 3 & -1 & 2\\ 1 & -2 & 3\\ 1 & -1 & 1\end{pmatrix}$

Solution:

$A = \begin{pmatrix}1 & 2 & -1\\ 3 & -1 & 2\\ 1 & -2 & 3\\ 1 & -1 & 1\end{pmatrix}$

$\sim \begin{pmatrix}1 & 2 & -1\\ 0 & -7 & 5\\ 0 & -4 & 4\\ 0 & -3 & 2\end{pmatrix}$ $R_2 \to R_2 - 3R_1;\ R_3 \to R_3 - R_1;\ R_4 \to R_4 - R_1$

$\sim \begin{pmatrix}1 & 2 & -1\\ 0 & -84 & 60\\ 0 & -84 & 84\\ 0 & -84 & 56\end{pmatrix}$ $R_2 \to 12R_2;\ R_3 \to 21R_3;\ R_4 \to 28R_4$

$\sim \begin{pmatrix}1 & 2 & -1\\ 0 & -84 & 60\\ 0 & 0 & 24\\ 0 & 0 & -4\end{pmatrix}$ $R_3 \to R_3 - R_2;\ R_4 \to R_4 - R_2$

$\sim \begin{pmatrix}1 & 2 & -1\\ 0 & -84 & 60\\ 0 & 0 & 4\\ 0 & 0 & -4\end{pmatrix}$ $R_3 \to \frac{1}{6}R_3$

$\sim \begin{pmatrix}1 & 2 & -1\\ 0 & -84 & 60\\ 0 & 0 & 4\\ 0 & 0 & 0\end{pmatrix}$ $R_4 \to R_4 + R_3$

This is in echelon form; no of nonzero rows = 3 $\Longrightarrow \rho(A) = 3$

Exercise 1.2 (2) (iii):

Find the rank of the matrices by row reduction method:

$$\begin{pmatrix} 3 & -8 & 5 & 2 \\ 2 & -5 & 1 & 4 \\ -1 & 2 & 3 & -2 \end{pmatrix}$$

Solution:

$$A = \begin{pmatrix} 3 & -8 & 5 & 2 \\ 2 & -5 & 1 & 4 \\ -1 & 2 & 3 & -2 \end{pmatrix} \sim \begin{bmatrix} -1 & 2 & 3 & -2 \\ 2 & -5 & 1 & 4 \\ 3 & -8 & 5 & 2 \end{bmatrix} R_1 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} -1 & 2 & 3 & -2 \\ 0 & -1 & 7 & 0 \\ 0 & -2 & 14 & -4 \end{bmatrix} R_2 \to R_2 + 2R_1;\ R_3 \to R_3 + 3R_1$$

$$\sim \begin{bmatrix} -1 & 2 & 3 & -2 \\ 0 & -1 & 7 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix} R_3 \to R_3 - 2R_2$$

$\Rightarrow \rho(A) = 3$

Exercise 1.2 (3)(i):

Find the inverse of the matrix by Gauss Jordan method:

$$\begin{pmatrix} 2 & -1 \\ 5 & -2 \end{pmatrix}$$

Solution: Let $A = \begin{pmatrix} 2 & -1 \\ 5 & -2 \end{pmatrix}$

$$[A|I] = \left(\begin{array}{cc|cc} 2 & -1 & 1 & 0 \\ 5 & -2 & 0 & 1 \end{array}\right) \qquad \text{l.. C. M. of 2 and 5 is 10}$$

$$\sim \left(\begin{array}{cc|cc} 10 & -5 & 5 & 0 \\ 10 & -4 & 0 & 2 \end{array}\right) R_1 \to 5R_1;\ R_2 \to 2R_2$$

$$\sim \left(\begin{array}{cc|cc} 10 & -5 & 5 & 0 \\ 0 & 1 & -5 & 2 \end{array}\right) R_2 \to R_2 - R_1$$

$$\sim \left(\begin{array}{cc|cc} 2 & -1 & 1 & 0 \\ 0 & 1 & -5 & 2 \end{array}\right) R_1 \to 5R_1$$

$$\sim \left(\begin{array}{cc|cc} 2 & 0 & -4 & 2 \\ 0 & 1 & -5 & 2 \end{array}\right) R_1 \to R_1 + R_2$$

$$\sim \left(\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & -5 & 2 \end{array}\right) R_1 \to \tfrac{1}{2}R_1$$

$\therefore\ A^{-1} = \begin{pmatrix} -2 & 1 \\ -5 & 2 \end{pmatrix}$

Exercise 1.3(1)(i):

Solve the following system of linear equations by matrix inversion method: $2x + 5y = -2,\ x + 2y = -3$

SOLUTION: $2x + 5y = -2,\ x + 2y = -3$

$$\begin{pmatrix} 2 & 5 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$$

$A \quad X = \quad B \quad \Rightarrow X = A^{-1}B$

$A = \begin{pmatrix} 2 & 5 \\ 1 & 2 \end{pmatrix}, X = \begin{pmatrix} x \\ y \end{pmatrix} B = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$

$|A| = \begin{vmatrix} 2 & 5 \\ 1 & 2 \end{vmatrix} = 4 - 5 = -1 \neq 0$, A^{-1} exists

adj $A = \begin{pmatrix} 2 & -5 \\ -1 & 2 \end{pmatrix}$

$A^{-1} = \frac{1}{|A|}(\text{adj } A) = \frac{1}{-1}\begin{pmatrix} 2 & -5 \\ -1 & 2 \end{pmatrix}$

$X = A^{-1}B$

$$\begin{pmatrix} x \\ y \end{pmatrix} = -1\begin{pmatrix} 2 & -5 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} -2 \\ -3 \end{pmatrix} = -1\begin{pmatrix} -4 + 15 \\ 2 - 6 \end{pmatrix} = -1\begin{pmatrix} 11 \\ -4 \end{pmatrix}$$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -11 \\ 4 \end{pmatrix} \Rightarrow x = -11,\ y = 4$

Exercise 1.3(1)(ii):

Solve the following system of linear equations by matrix inversion method: $2x - y = 8,\ 3x + 2y = -2$

SOLUTION: $2x - y = 8,\ 3x + 2y = -2$

$$\begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ -2 \end{pmatrix} \Rightarrow A X = B \Rightarrow X = A^{-1}B$$

$A = \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}, X = \begin{pmatrix} x \\ y \end{pmatrix}, B = \begin{pmatrix} 8 \\ -2 \end{pmatrix}$

$|A| = \begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} = 4 + 3 = 7 \neq 0$, A^{-1} exists

adj $A = \begin{pmatrix} 2 & 1 \\ -3 & 2 \end{pmatrix} \Rightarrow A^{-1} = \frac{1}{|A|}(\text{adj } A) = \frac{1}{7}\begin{pmatrix} 2 & 1 \\ -3 & 2 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 2 & 1 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 8 \\ -2 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 16 - 2 \\ -24 - 4 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 14 \\ -28 \end{pmatrix} = \begin{pmatrix} \frac{14}{7} \\ \frac{-28}{7} \end{pmatrix}$$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix} \quad \Rightarrow\ x = 2,\quad y = -4$

Exercise 1.3(4):

Four men and 4 women can finish a piece of work jointly in 3 days while 2 men and 5 women can finish the same work jointly in 4 days. Find the time taken by one man alone and that of one woman alone to finish the same work by using matrix inversion method.

SOLUTION:

Let one man complete the work in x days

let one women complete the work in y days

man one day work $= \frac{1}{x}$, women one day work $= \frac{1}{y}$

Given: $4(\frac{1}{x}) + 4(\frac{1}{y}) = \frac{1}{3},\ 2(\frac{1}{x}) + 5(\frac{1}{y}) = \frac{1}{4}$

$$\begin{pmatrix} 4 & 4 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} \frac{1}{x} \\ \frac{1}{y} \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{4} \end{pmatrix} \Rightarrow A X = B \Rightarrow X = A^{-1}B$$

$A = \begin{pmatrix} 4 & 4 \\ 2 & 5 \end{pmatrix}, X = \begin{pmatrix} \frac{1}{x} \\ \frac{1}{y} \end{pmatrix};\ B = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{4} \end{pmatrix}$

$|A| = \begin{vmatrix} 4 & 4 \\ 2 & 5 \end{vmatrix} = 20 - 8 = 12 \neq 0$, A^{-1} exists

adj $A = \begin{pmatrix} 5 & -4 \\ -2 & 4 \end{pmatrix} \Rightarrow A^{-1} = \frac{1}{12}\begin{pmatrix} 5 & -4 \\ -2 & 4 \end{pmatrix}$

$$X = \frac{1}{12}\begin{pmatrix} 5 & -4 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} \frac{1}{3} \\ \frac{1}{4} \end{pmatrix} = \frac{1}{12}\begin{pmatrix} \frac{5}{3} + \frac{-4}{4} \\ \frac{-2}{3} + \frac{4}{4} \end{pmatrix} = \frac{1}{12}\begin{pmatrix} \frac{20-12}{12} \\ \frac{-8+12}{12} \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{x} \\ \frac{1}{y} \end{pmatrix} = \frac{1}{12}\begin{pmatrix} \frac{8}{12} \\ \frac{4}{12} \end{pmatrix} = \begin{pmatrix} \frac{8}{144} \\ \frac{4}{144} \end{pmatrix} = \begin{pmatrix} \frac{1}{18} \\ \frac{1}{36} \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{x} \\ \frac{1}{y} \end{pmatrix} = \begin{pmatrix} \frac{1}{18} \\ \frac{1}{36} \end{pmatrix} \quad \Rightarrow \frac{1}{x} = \frac{1}{18} \Rightarrow x = 18 \text{ and } \frac{1}{y} = \frac{1}{36} \Rightarrow\ y = 36$$

One man can complete the work in 18 days

one woman can complete the work in 36 days

Exercise 1.3(3):

A man is appointed in a job with a monthly salary of certain amount and a fixed amount of annual increment. If his salary was ₹19,800 per month at the end of the first month after 3 years of service and ₹23,400 per month at the end of the first month after 9 years of service, find his starting salary and his annual increment.

(Use matrix inversion method to solve the problem.)

SOLUTION:

Let salary be ₹ x and annual increment be ₹ y

Given: $x + 3y = 19800$ and $x + 9y = 23400$

$\begin{pmatrix} 1 & 3 \\ 1 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 19800 \\ 23400 \end{pmatrix}$

$AX = B \Rightarrow X = A^{-1} B$ and $A^{-1} = \frac{1}{|A|}(\text{adj } A)$

$|A| = \begin{vmatrix} 1 & 3 \\ 1 & 9 \end{vmatrix} = 9 - 3 = 6 \neq 0$, A^{-1} exists

adj A $= \begin{pmatrix} 9 & -3 \\ -1 & 1 \end{pmatrix}$ & $A^{-1} = \frac{1}{6}\begin{pmatrix} 9 & -3 \\ -1 & 1 \end{pmatrix}$

$X = \frac{1}{6}\begin{pmatrix} 9 & -3 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 19800 \\ 23400 \end{pmatrix}$

$= \frac{1}{6}\begin{pmatrix} 178200 - 70200 \\ -19800 + 23400 \end{pmatrix}$

$= \frac{1}{6}\begin{pmatrix} 108000 \\ 3600 \end{pmatrix}$

$= \begin{pmatrix} 108000/6 \\ 3600/6 \end{pmatrix} = \begin{pmatrix} 18000 \\ 600 \end{pmatrix}$

Initial salary x = ₹ 18000, annual increment = ₹ 600

Exercise 1.4(1)(i):

Solve : $5x - 2y + 16 = 0, x + 3y - 7 = 0$

Solution: $5x - 2y + 16 = 0, x + 3y - 7 = 0$

$5x - 2y = -16$, $x + 3y = 7$

$\Delta = \begin{vmatrix} 5 & -2 \\ 1 & 3 \end{vmatrix} = 15 + 2 = 17$

$\Delta_x = \begin{vmatrix} -16 & -2 \\ 7 & 3 \end{vmatrix} = -48 + 14 = -34$

$\Delta_y = \begin{vmatrix} 5 & -16 \\ 1 & 7 \end{vmatrix} = 35 + 16 = 51$

$x = \frac{\Delta_x}{\Delta} = \frac{-34}{17} = -2$, $x = -2$

$y = \frac{\Delta_y}{\Delta} = \frac{51}{17} = 3$, $y = 3$

Exercise 1.4(1)(ii): Solve the following systems of linear equations by Cramer's rule: $\frac{3}{x} + 2y = 12$, $\frac{2}{x} + 3y = 13$

Solution:

$\Delta = \begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix} = 9 - 4 = 5$

$\Delta_{\frac{1}{x}} = \begin{vmatrix} 12 & 2 \\ 13 & 3 \end{vmatrix} = 36 - 26 = 10$

$\Delta_y = \begin{vmatrix} 3 & 12 \\ 2 & 13 \end{vmatrix} = 39 - 24 = 15$

$\frac{1}{x} = \frac{10}{5} = 2$, $y = \frac{15}{5} = 3$

$x = \frac{1}{2}$, $y = 3$

Exercise 1.4(2):

In a competitive examination, one mark is awarded for every correct answer while $\frac{1}{4}$ mark is deducted for every wrong answer. A student answered 100 questions and got 80 marks. How many questions did he answer correctly ? (Use Cramer's rule to solve the problem).

Solution: Let Number of question answered correctly be x

Let Number of question answered wrong be y

Given : For correct answer 1 mark, wrong answer $-\frac{1}{4}$ mark

$x + y = 100$; $x - \frac{1}{4}y = 80 \Rightarrow 4x - y = 320$

$\Delta = \begin{vmatrix} 1 & 1 \\ 4 & -1 \end{vmatrix} = -1 - 4 = -5$

$\Delta_x = \begin{vmatrix} 100 & 1 \\ 320 & -1 \end{vmatrix} = -100 - 320 = -420$

$\Delta_y = \begin{vmatrix} 1 & 100 \\ 4 & 320 \end{vmatrix} = 320 - 400 = -80$

$X = \frac{\Delta_x}{\Delta} = \frac{-420}{-5} = 84$, No. of questions answered correctly = 84

$Y = \frac{\Delta_y}{\Delta} = \frac{-80}{-5} = 16$. No. of question answered wrong = 16

Exercise 1.4(3):

A chemist has one solution which is 50% acid and another solution which is 25% acid. How much each should be mixed to make 10 litres of a 40% acid solution? (Use Cramer's Rule)

Solution: Let 50% acid be x litres and 25% acid be y litres

Given: $x + y = 10$

$\frac{50}{100}x + \frac{25}{100}y = \frac{40}{100}(10) \Rightarrow 10x + 5y = 80$

$x + y = 10$; $10x + 5y = 80$

$\Delta = \begin{vmatrix} 1 & 1 \\ 10 & 5 \end{vmatrix} = 5 - 10 = -5$

$\Delta_x = \begin{vmatrix} 10 & 1 \\ 80 & 5 \end{vmatrix} = 50 - 80 = -30$

$\Delta_y = \begin{vmatrix} 1 & 10 \\ 10 & 80 \end{vmatrix} = 80 - 100 = -20$

$x = \frac{\Delta_x}{\Delta} = \frac{-30}{-5} = 6$, 50% acid 6 litres to be mixed

$y = \frac{\Delta_y}{\Delta} = \frac{-20}{-5} = 4$, 25% acid 4 litres to be mixed

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Exerscise 1.4(4):

A fish tank can be filled in 10 minutes using both pumps A and B simultaneously. However, Pump B can pump water in or out at the same rate. If Pump B is dvertently run in reverse, then the tank will be filled in 30 minutes. How long would it take each pump to fill the tank by it self

SOLUTION:

Let the Pump A and Pump B fill the tank in x and y mins.

Water filled by Pump A and Pump B in 1 min is $\frac{1}{x}, \frac{1}{y}$ resp.

Given: $\frac{1}{x} + \frac{1}{y} = \frac{1}{10}$ and $\frac{1}{x} - \frac{1}{y} = \frac{1}{30}$

$\Rightarrow \frac{10}{x} + \frac{10}{y} = 1$ and $\frac{30}{x} - \frac{30}{y} = 1$

$\Delta = \begin{vmatrix} 10 & 10 \\ 30 & -30 \end{vmatrix} = -300 - 300 = -600$

$\Delta_{\frac{1}{x}} = \begin{vmatrix} 1 & 10 \\ 1 & -30 \end{vmatrix} = -30 - 10 = -40$

$\Delta_{\frac{1}{y}} = \begin{vmatrix} 10 & 1 \\ 30 & 1 \end{vmatrix} = 10 - 30 = -20$

$\frac{1}{x} = \frac{-40}{-600} = \frac{1}{15}$, Pump A fill the tank in 15 mins

$\frac{1}{y} = \frac{-20}{-600} = \frac{1}{30}$, Pump B fill the tank in 30 mins

Exercise 1.6(1)(iii):
Test for consistency and if possible, solve the following systems of equations by rank method:
$2x + 2y + z = 5, x - y + z = 1, 3x + y + 2z = 4$
Solution:
$$\begin{pmatrix} 2 & 2 & 1 \\ 1 & -1 & 1 \\ 3 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 4 \end{pmatrix}$$
A X = B
$$[A|B] = \begin{bmatrix} 2 & 2 & 1 & 5 \\ 1 & -1 & 1 & 1 \\ 3 & 1 & 2 & 4 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & -1 & 1 & 1 \\ 2 & 2 & 1 & 5 \\ 3 & 1 & 2 & 4 \end{bmatrix} R_1 \leftrightarrow R_2$$
$$\sim \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 4 & -1 & 3 \\ 0 & 4 & -1 & 1 \end{bmatrix} R_2 \to R_2 - 2R_1 ; R_3 \to R_3 - 3R_1$$
$$\sim \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 4 & -1 & 3 \\ 0 & 0 & 0 & -2 \end{bmatrix} R_3 \to R_3 - R_2$$
$\rho([A|B]) = 3 , \rho(A) = 2$

$\rho([A|B]) \neq \rho(A)$
System inconsistent , No solution

Exercise 1.7(1)(ii):
Solve the following system of homogenous equations:
$2x + 3y - z = 0, x - y - 2z = 0, 3x + y + 3z = 0$
Solution:
$2x + 3y - z = 0, x - y - 2z = 0, 3x + y + 3z = 0$
$$\begin{pmatrix} 2 & 3 & -1 \\ 1 & -1 & -2 \\ 3 & 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$
A X = O
$$[A|O] = \begin{bmatrix} 2 & 3 & -1 & 0 \\ 1 & -1 & -2 & 0 \\ 3 & 1 & 3 & 0 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & -1 & -2 & 0 \\ 2 & 3 & -1 & 0 \\ 3 & 1 & 3 & 0 \end{bmatrix} R_1 \leftrightarrow R_2$$
$$\sim \begin{bmatrix} 1 & -1 & -2 & 0 \\ 0 & 5 & 3 & 0 \\ 0 & 4 & 9 & 0 \end{bmatrix} R_2 \to R_2 - 2 R_1; R_3 \to R_3 - 3 R_1$$
$$\sim \begin{bmatrix} 1 & -1 & -2 & 0 \\ 0 & 20 & 12 & 0 \\ 0 & 20 & 45 & 0 \end{bmatrix} R_2 \to 4R_2 ; R_3 \to 5 R_3$$
$$\sim \begin{bmatrix} 1 & -1 & -2 & 0 \\ 0 & 20 & 12 & 0 \\ 0 & 0 & 33 & 0 \end{bmatrix} R_3 \to R_3 - R_2$$
$\Rightarrow$ $\rho([A|O]) = 3 , \rho(A) = 3$
$\Rightarrow$ $\rho([A| O]) = \rho(A) = 3 =$ Number of unknowns
$\Rightarrow$ system consistent with unique solution
$\Rightarrow$ System consistent with trivial solution

$\Rightarrow$ $x = 0, y = 0, z = 0$

Question 15.
Decrypt the received encoded message $[2 \quad -3][20 \quad 4]$ with the encryption matrix $\begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$ and the decryption matrix as its inverse, where the system of codes are described by the numbers $1 - 26$ to the letters A- Z respectively, and the number 0 to a blank space.
Solution:
Let the encoding matrix be $\begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$

Let $A = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}$

$|A| = \begin{vmatrix} -1 & -1 \\ 2 & 1 \end{vmatrix} = -1 + 2 = 1$

Now adj $A = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}$

So $A^{-1} = \frac{1}{1}\begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}$

Now coded row matrix (B)	Decoded row matrix (BA^{-1})
$(2 \quad -3)$	$\overrightarrow{(2 \quad -3)}\begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} = (2+6 \;\; 2+3)$ $=(8 \quad 5)$
$(20 \quad 4)$	$(20 \quad 4)\begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}$ $= (20-8 \;\; 20-4)$ $= (12 \quad 16)$

So the sequence of decoded matrices is [8 5] , [12 16]. Thus the receivers read this message as HELP.

CHAPTER 2 : COMPLEX NUMBERS

2 MARKS, 3 MARKS, 5 MARKS

2 MARKS

Ex 2.1

Simplify:

(i) $i^{1947}+i^{1950}$ $\quad 1947 = 1944+3$

$= i^{1944}\cdot i^3 + i^{1948}\cdot i^2$ $\quad 1950 = 1948+2$

$= i^3+i^2 = -i-1$ $\quad i^{1944}=1$

$= -1-i$ $\quad i^{1948}=1$

---

2. $i^{1948}-i^{-1869}$ $\quad 1948 =$ multiple of 4

$= i^{1948}-\frac{1}{i^{1869}}$ $\quad 1869 = 1868+1$

$= 1-\frac{1}{i^{1869}} = 1-\frac{1}{i^{1868+1}} = 1-\frac{1}{i^{1868}\cdot i^1}$

$= 1-\frac{1}{i} = 1-\frac{1}{i}\times\frac{i}{i} = 1-\frac{i}{i^2} = 1-(-i) = 1+i$

---

3. $\sum_{n=1}^{12} \mathbf{i^{12} = i^1+i^2+i^3+i^4+i^5+i^6+i^7+i^8+i^9+i^{10}+i^{11}+i^{12}}$

$= i-1-i+1+i.i^4+i^4i^2+i^4i^3+i^4i^4+i^8i+i^8i^2+i^8i^3+i^8i^4$

$= i-1-i+1+i-1-i+1+i-1-i+1 = 0$

---

4. $\mathbf{i^{59}+\frac{1}{i^{59}} = i^{56}\cdot i^3+\frac{1}{i^{56}i^3}}$ $\quad \left(\because \frac{1}{i^3}=i\right)$

$\mathbf{= -i+\frac{1}{1\cdot(i)^3} = -i+i = 0}$

---

5. $\mathbf{i\,.\,i^2.\,i^3\ldots i^{2000} = i^{1+2+3+\cdots+2000}}$ $\quad \sum \mathbf{n} = \frac{\mathbf{n(n+1)}}{\mathbf{2}}$

$\mathbf{= i^{\frac{2000(2000+1)}{2}} = i^{1000\times2001} = 1}$ $\quad [\because \mathbf{i^{(\text{multiple of }4)}) = 1.}]$

---

6. $\sum_{n=1}^{10} \mathbf{i^{n+50}}$ $\quad \mathbf{i^n+i^{n+1}+i^{n+2}+i^{n+3} = 0}$

$= (i^{51}+i^{52}+i^{53}+i^{54})+(i^{55}+i^{56}+i^{57}+i^{58}+i^{59}+i^{60})$

$= 0+0+i^{56}\cdot i^3+1 = -i+1 = 1-i$

Ex 2.2

(1) $z = 5-2i$ $\quad w = -1+3i$ Find the value of

(i) $z+w = 5-2i+(-1+3i) = 5-2i-1+3i = 4+i$

(ii) $z-iw = 5-2i-i(-1+3i) = 5-2i+i-3i^2$

$= 5-i-3(-1) = 5-i+3 = 8-i$

(iii) $2z+3w = 2(5-2i)+3(-1+3i) = 10-4i-3+9i$

$= 7+5i$

(iv) $zw = (5-2i)(-1+3i) = -5+15i+2i-6i^2$

$= -5+17i+6 = 1+17i$

(v) $z^2+2zw+w^2 = (z+w)^2 = (4+i)^2$ (Ref(i))

$= 4^2+2(4)i+i^2 = 16+8i-1 = 15+8i$

(vi) $(z+w)^2 = (4+i)^2 = 16+8i-1 = 15+8i$

EXERCISE 2.3

1. $\mathbf{z_1 = 1-3i}$, $\mathbf{z_2 = -4i}$, $\mathbf{z_3 = 5}$

(i) $\mathbf{(z_1+z_2)+z_3 = z_1+(z_2+z_3)}$

L.H.S $= (z_1+z_2)+z_3$

$= [1-3i+(-4i)]+5 = (1-3i-4i)+5$

$= 1-7i+5 = 6-7i$ $\quad$ - (1)

R.H.S $= z_1+(z_2+z_3)$

$= 1-3i+(-4i+5)$

$= 1-3i-4i+5$

$= 6-7i$ $\quad$ - (2)

(1) = (2) $\quad$ LHS = RHS

$\therefore (z_1+z_2)+z_3 = z_1+(z_2+z_3)$

(ii) $\mathbf{(z_1z_2)z_3 = z_1(z_2z_3)}$

L.H.S : $z_1z_2 = (1-3i)(-4i) = -4i+12i^2 = -4i-12$

$(z_1z_2)z_3 = (-12-4i)5 = -60-20i$ $\quad$ - (3)

R.H.S : $z_2z_3 = (-4i)5 = -20i$

$z_1(z_2z_3) = (1-3i)(-20i) = -20i+60i^2$

$= -60-20i$ $\quad$ - (4)

(3) =(4) $\quad (z_1z_2)z_3 = z_1(z_2z_3)$

EXERCISE 2.4

1. **Write in the rectangular form.**

(1) $\overline{(5+9i)+(2-4i)}$

$= \overline{5+9i}+\overline{2-4i} = 5-9i+2+4i = 7-5i$

(ii) $\frac{10-5i}{6+2i} = \frac{10-5i}{6+2i}\times\frac{6-2i}{6-2i} = \frac{60-20i-30i+10i^2}{6^2+2^2}$

$= \frac{60-50i-10}{36+4} = \frac{50-50i}{40} = \frac{10(5-5i)}{40} = \frac{5}{4}-\frac{5i}{4} = \frac{\mathbf{5(1-i)}}{\mathbf{4}}$

(iii) $3\bar{i}+\frac{1}{2-i} = -3i+\frac{1}{2-i}\times\frac{2+i}{2+i} = -3i+\frac{2+i}{2^2+1^2}$

$= -3i+\frac{2+i}{5} = \frac{-15i+2+i}{5} = \frac{2-14i}{5} = \frac{\mathbf{2}}{\mathbf{5}}\mathbf{(1-7i)}$

---

(2) **Find the rectangular form of the following** $\quad \mathbf{z = x+iy.}$

(i) $\mathbf{Re}\left(\frac{1}{z}\right)$ $\quad z = x+iy$

$\frac{1}{z} = z^{-1} = \frac{x}{x^2+y^2}+i\frac{-y}{x^2+y^2}$ $\quad \therefore \mathbf{Re}\left(\frac{1}{z}\right) = \frac{\mathbf{x}}{\mathbf{x^2+y^2}}$

(ii) $\mathbf{Re\,(i\bar{z})}$

$z = x+iy$ $\quad \therefore \bar{z} = x-iy$

$i\bar{z} = i(x-iy) = ix-i^2y = y+ix$ $\quad \therefore \mathbf{Re\,(i\bar{z}) = y}$

(iii) $\mathbf{Im}\ (\mathbf{3z+4\bar{z}-4i})$

$$3z+4\bar{z}-4i = 3(x+iy)+4(x-iy)-4i$$
$$= 3x+i3y+4x-i4y-4i$$
$$= (3x+4x)+i(3y-4y-4)$$
$$= 7x+i(-y-4)$$
$$\mathrm{Im}(3z+4\bar{z}-4i) = -y-4$$

(3) **If $\mathbf{z_1=2-i}$, $\mathbf{z_2=-4+3i}$, Find the inverses of $\mathbf{z_1z_2}$ & $\frac{\mathbf{z_1}}{\mathbf{z_2}}$.**

Solution :

$$z_1z_2 = (2-i)(-4+3i) = -8+6i+4i-3i^2$$
$$= -8+10i+3 = -5+10i$$
$$(z_1z_2)^{-1} = \frac{-5}{(-5)^2+10^2}+i\frac{-10}{(-5)^2+10^2} = \frac{-5-10i}{25+100} = \frac{-5(1+2i)}{125}$$
$$= \frac{1}{25}(-1-2i)$$
$$\left(\frac{z_1}{z_2}\right)^{-1} = \frac{z_2}{z_1} = \frac{-4+3i}{2-i}\times\frac{2+i}{2+1} = \frac{-8-4i+6i+3i^2}{2^2+1^2}$$
$$= \frac{-8+2i-3}{4+1} = \frac{1}{5}(-11+2i)$$

EXERCISE 2.5

1. (i) $\left|\frac{2i}{3+4i}\right| = \frac{|2i|}{|3+4i|} = \frac{|2||i|}{\sqrt{3^2+4^2}} = \frac{2(1)}{\sqrt{25}} = \frac{2}{5}$

1 (ii) $\left|\frac{2-i}{1+i}+\frac{1-2i}{1-i}\right| = \left|\frac{(2-i)(1-i)+(1-2i)(1+i)}{(1+i)(1-i)}\right|$

$$= \left|\frac{2-2i-i+i^2+1+i-2i-2i^2}{1^2+1^2}\right|$$
$$= \left|\frac{2-3i-1+1-i+2}{1+1}\right| = \left|\frac{4-4i}{2}\right| = \frac{\sqrt{4^2+(-4)^2}}{2}$$
$$= \frac{\sqrt{32}}{2} = \frac{4\sqrt{2}}{2} = 2\sqrt{2}$$

EXERCISE 2.6

(3) Obtain cartesian form of the locus of $\ = \mathbf{x+iy}$

(i) $[\mathbf{Re}\ (\mathbf{iz})]^2 = \mathbf{3}$

Solution :

$\mathbf{z=x+iy}$

$iz = i(x+iy) = ix+i^2y \ = -y+ix$

$\mathrm{Re}\ (iz) = -y$

$[\mathrm{Re}(iz)]^2 = (-y)^2 = y^2$

$\therefore [\mathbf{Re(iz)}]^2 = \mathbf{3} \Rightarrow \mathbf{y^2=3}$

(ii) $\mathbf{Im}\ [(\mathbf{1-i})\mathbf{z}+\mathbf{1}] = \mathbf{0}$.

Soln: $\mathbf{z=x+iy}$

$$(1-i)z+1 = (1-i)(x+iy)+1$$
$$= x+iy-ix-i^2y+1$$
$$= x+iy-ix+y+1$$
$$= (x+y+1)+i(y-x)$$

$\mathbf{Im}\ [(\mathbf{1-i})\mathbf{z+1}] = \mathbf{0} \Rightarrow \mathbf{y-x=0} \Rightarrow \mathbf{x=y}$

(iii) $|\mathbf{z+i}| = |\mathbf{z-1}|$ $\quad [z=x+iy]$

$$|x+iy+i| = |x+iy-1|$$
$$|x+i(y+1)| = |x-1+iy|$$
$$\Rightarrow \sqrt{x^2+(y+1)^2} = \sqrt{(x-1)^2+y^2}$$
$$\Rightarrow x^2+(y+1)^2 = (x-1)^2+y^2$$
$$\Rightarrow x^2+y^2+2y+1 = x^2-2x+1+y^2 \Rightarrow 2x+2y=0$$

$\Rightarrow x+y=0$ **Locus of z is $\mathbf{x+y=0}$**

(iv) $\bar{\mathbf{z}} = \mathbf{z^{-1}} = \frac{\mathbf{1}}{\mathbf{z}}$

$$\Rightarrow \mathbf{z\bar{z}=1} \Rightarrow |\mathbf{z}|^2=\mathbf{1} \Rightarrow |\mathbf{x+iy}|^2=\mathbf{1} \Rightarrow \sqrt{\mathbf{x^2+y^2}}=\mathbf{1}$$
$$\Rightarrow \mathbf{x^2+y^2=1}$$

(4) show that the following eqns represent a circle , and find its centre and radius . (each 2 Mark)

(i) $|\mathbf{z-2-i}| = \mathbf{3} \Rightarrow |\mathbf{z-(2+i)}| = \mathbf{3}$

It is in the form of $|\mathbf{z-z_0}| = \mathbf{a}$; it forms or rep eqn of circle

$\mathbf{z_0=2+i}$ **i.e $(\mathbf{2,1})$ $\quad \mathbf{a=3}$**

(ii) $|\mathbf{2z+2-4i}| = \mathbf{2}$

$\div \mathbf{2} \quad |\mathbf{z+1-2i}| = \mathbf{1} \Rightarrow |\mathbf{z-(-1+2i)}| = \mathbf{1}$

It is in the form of $|\mathbf{z-z_0}| = \mathbf{a}$; it forms or rep eqn of circle

$\mathbf{z_0=-1+2i}$ **i.e. $(-\mathbf{1},\mathbf{2})$ $\quad \mathbf{a=1}$**

(ii) $|\mathbf{3z-6+12i}| = \mathbf{8}$

$\div \mathbf{3} \quad |\mathbf{z-2+4i}| = \frac{\mathbf{8}}{\mathbf{3}} \quad \Rightarrow |\mathbf{z-(+2-4i)} = \frac{\mathbf{8}}{\mathbf{3}}$

It is in the form of $|\mathbf{z-z_0}| = \mathbf{a}$; it forms or rep eqn of circle

center $\mathbf{z_0=2-4i}$ i.e $(\mathbf{2},-\mathbf{4})$ $\quad \mathbf{a}=\frac{\mathbf{8}}{\mathbf{3}}$

5. Obtain the cartesian eqn for the locus of

$\mathbf{z=x+iy}$ in each of the following cases.

(i) $|\mathbf{z-4}| = \mathbf{16}$ $\quad \mathbf{z=x+iy}$

$$|\mathbf{x+iy-4}| = \mathbf{16} \Rightarrow |\mathbf{x-4+iy}| = \mathbf{16}$$
$$\sqrt{(\mathbf{x-4})^2+\mathbf{y^2}} = \mathbf{16}$$
$$\mathbf{x^2-8x+16+y^2 = 16^2 = 256}$$
$$\mathbf{x^2+y^2-8x+16-256=0} \Rightarrow \mathbf{x^2+y^2-8x-240=0}$$

(ii) $|\mathbf{z-4}|^2-|\mathbf{z-1}|^2 = \mathbf{16}$ Given $\mathbf{z=x+iy}$

$$|\mathbf{x+iy-4}|^2-|\mathbf{x+iy-1}|^2 = \mathbf{16}$$
$$|(\mathbf{x-4})+\mathbf{iy}|^2-|(\mathbf{x-1})+\mathbf{iy}|^2 = \mathbf{16}$$
$$\left[\sqrt{(\mathbf{x-4})^2+\mathbf{y^2}}\right]^2-\left[\sqrt{(\mathbf{x-1})^2+\mathbf{y^2}}\right]^2 = \mathbf{16}$$
$$(\mathbf{x-4})^2+\mathbf{y^2}-[(\mathbf{x-1})^2+\mathbf{y^2}] = \mathbf{16}$$
$$\mathbf{x^2-8x+16+y^2-(x^2-2x+1+y^2) = 16}$$
$$\mathbf{x^2-8x+16+y^2-x^2+2x-1-y^2 = 16}$$
$$\mathbf{-8x+16+2x-1-16=0}$$
$$\mathbf{-6x-1=0} \Rightarrow \mathbf{-6x=1}$$

Locus of z ix $\mathbf{x}=\frac{-\mathbf{1}}{\mathbf{6}}$ or $\mathbf{6x+1=0}$

EXERCISE 2.7

1. Write the polar form.

(i) $\mathbf{2 + i\,2\sqrt{3} = r(\cos\theta + i\sin\theta)}$ $\quad \mathbf{a = 2} \quad \mathbf{b = 2\sqrt{3}}$

$r = \sqrt{a^2 + b^2} = \sqrt{2^2 + (2\sqrt{3})^2} = \sqrt{4 + 12} = \sqrt{16} = 4$

$\alpha = \tan^{-1}\left|\frac{b}{a}\right| = \tan^{-1}\left|\frac{2\sqrt{3}}{2}\right| = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$

$2 + 2\sqrt{3}\,\text{I}$ lies in I quadrant $\therefore \theta = \alpha = \frac{\pi}{3}$

$\therefore 2 + i2\sqrt{3} = 4\left[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right]$

General form

$\mathbf{2 + i2\sqrt{3} = 4\left[\cos\left(2k\pi + \frac{\pi}{3}\right) + i\sin\left(2k\pi + \frac{\pi}{3}\right)\right. \quad k \in z}$

(ii) $\mathbf{3 - i\sqrt{3} = r(\cos\theta + i\sin\theta)}$ $\quad \mathbf{a = 3} \quad \mathbf{b = -\sqrt{3}}$

$r = \sqrt{a^2 + b^2} = \sqrt{3^2 + (-\sqrt{3})^2} = \sqrt{9 + 3} = \sqrt{12} = 2\sqrt{3}$

$\alpha = \tan^{-1}\left|\frac{b}{a}\right| = \tan^{-1}\left|-\frac{\sqrt{3}}{3}\right| = \tan^{-1}\left|-\frac{\sqrt{3}}{\sqrt{3}\sqrt{3}}\right|$

$= \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

$\theta = -\alpha = -\frac{\pi}{6}$ $\begin{pmatrix} 3 - i\sqrt{3} \text{ lies in} \\ \text{IV quadrant} \end{pmatrix}$

$3 - i\sqrt{3} = 2\sqrt{3}\left[\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right]$

$= 2\sqrt{3}\left[\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right]$

$\mathbf{3 - i\sqrt{3} = 2\sqrt{3}\left[\cos\left(2k\pi + \frac{\pi}{6}\right) - i\sin\left(2k\pi + \frac{\pi}{6}\right)\right] k \in z}$

(ii) $\mathbf{-2 - i2 = r(\cos\theta + i\sin\theta)}$

$a = -2 \quad b = -2$

$r = \sqrt{(-2)^2 + (-2)^2} = \sqrt{4 + 4} = \sqrt{8} = 2\sqrt{2}$

$\alpha = \tan^{-1}\left|\frac{b}{a}\right| = \tan\left|\frac{-2}{-2}\right| = \tan^{-1}(1) = \frac{\pi}{4}$

$\theta = -\pi + \alpha = -\pi + \frac{\pi}{4}$ $\quad (-2 - i2$ lies in III quadrant $)$

$= \frac{-4\pi + \pi}{4} = \frac{-3\pi}{4}$

$-2 - i2 = 2\sqrt{2}\left[\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right]$

$\mathbf{-2 - i2 = 2\sqrt{2}\left[\cos\left(2k\pi - \frac{3\pi}{4}\right) + i\sin\left(2k\pi - \frac{3\pi}{4}\right)\right] \quad , k \in z}$

EXERCISE 2.8

1. If $\mathbf{\omega \neq 1}$ **; S.T.** $\mathbf{\frac{a+b\omega+c\omega^2}{b+c\omega+a\omega^2} + \frac{a+b\omega+c\omega^2}{c+a\omega+b\omega^2} = -1}$

L.H.S$= \mathbf{\frac{a+b\omega+c\omega^2}{b+c\omega+a\omega^2} \times \frac{\omega}{\omega} + \frac{a+b\omega+c\omega^2}{c+a\omega+b\omega^2} \times \frac{\omega^2}{\omega^2}}$

$= \mathbf{\frac{(a+b\omega+c\omega^2)\omega}{b\omega+c\omega^2+a\omega^3} + \frac{a+b\omega+c\omega^2}{c\omega^2+a\omega^3+b\omega^2} \times \omega^2}$

$= \mathbf{\frac{(a+b\omega+c\omega^2)\omega}{a+b\omega+c\omega^2} + \frac{a+b\omega+c\omega^2}{c\omega^2+a+b\omega} \times \omega^2 \quad = \omega + \omega^2 = -1}$

3 MARKS

EXERCISE 2.2

(2) Given the complex number $\mathbf{z = 2 + 3i}$**, represent the complex number in Argand diagram**

(i) $z, iz, z + iz$

$z = 2 + 3i$

$iz = i(2 + 3i) = 2i + 3i^2 = 2i + 3(-1) = -3 + 2i$

$z + iz = 2 + 3i - 3 + 2i = -1 + 5i$

(ii) $z = 2 + 3i \quad z, -iz, z - iz$

$z = 2 + 3i$

$-iz = -i(2 + 3i) = -2i - 3i^2 = -2i + 3 = 3 - 2i$

$z - iz = z + (-iz) = 2 + 3i + 3 - 2i = 5 + i$

(3) **Find x and y**

$\mathbf{(3 - i)x - (2 - i)y + 2i + 5 = 2x + (-1 + 2i)y + 3 + 2i}$

$3x - ix - 2y + yi + 2i + 5 = 2x - y + i2y + 3 + 2i$

$(3x - 2y + 5) + i(-x + y + 2) = (2x - y + 3) + i(2y + 2)$

Eqn Real $3x - 2y + 5 = 2x - y + 3$

$3x - 2y - 2x + y = 3 - 5$

$x - y = -2 \quad - (1)$

Eqn Img $\quad -x + y + 2 = 2y + 2$

$-x + y - 2y = 2 - 2$

$-x - y = 0 \quad - (2)$

$x - y = -2$

$-x - y = 0$

$-2y = -2$

$y = 1$

$y = 1 \quad -x - 1 = 0$

$-x = 1$

$x = -1, \quad y = 1$

EXERCISE 2.3

1. $\mathbf{z_1 = 1 - 3i,\ z_2 = -4i,\ z_3 = 5}$

(i) **S.T.** $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$

L.H.S $= (z_1 + z_2) + z_3$

$= [1 - 3i + (-4i)] + 5 = (1 - 3i - 4i) + 5$

$= 1 - 7i + 5 = 6 - 7i \quad - (1)$

R.H.S $= z_1 + (z_2 + z_3) \quad = 1 - 3i + (-4i + 5)$

$= 1 - 3i - 4i + 5 \quad = 6 - 7i \quad - (2)$

$(1) = (2)$

LHS = RHS

$\therefore \mathbf{(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)}$

(ii) $\mathbf{(z_1z_2)z_3 = z_1(z_2z_3)}$

L.H.S : $z_1z_2 = (1-3i)(-4i) = -4i + 12i^2 = -4i - 12$

$(z_1z_2)z_3 = (-12-4i)5 = -60 - 20i$ - (3)

R.H.S : $z_2z_3 = (-4i)5 = -20i$

$z_1(z_2z_3) = (1-3i)(-20i) = -20i + 60i^2$

$= -60 - 20i$ - (4)

(3) =(4) $\mathbf{(z_1z_2)z_3 = z_1(z_2z_3)}$

---

(2) $\mathbf{z_1 = 3 \quad z_2 = -7i \quad z_3 = 5 + 4i}$

(i) $\mathbf{z_1(z_2 + z_3) = z_1z_2 + z_1z_3}$

L.H.S : $z_2 + z_3 = -7i + 5 + 4i = 5 - 3i$

$z_1(z_2 + z_3) = 3(5-3i) = 15 - 9i$ - (1)

R.H.S : $z_1z_2 = 3(-7i) = -21i$

$z_1z_3 = 3(5+4i) = 15 + 12i$

$z_1z_2 + z_1z_3 = -21i + 15 + 12i$

$= 15 - 9i$ - (2)

(1) = (2) $\mathbf{z_1(z_2 + z_3) = z_1z_2 + z_1z_3}$

(ii) $\mathbf{(z_1 + z_2)z_3 = z_1z_3 + z_2z_3}$

L.H.S : $z_1 + z_2 = 3 + (-7i) = 3 - 7i$

$(z_1 + z_2)z_3 = (3-7i)(5+4i)$

$= 15 + 12i - 35i - 28i^2$

$= 15 - 23i + 28 = 43 - 23i$ - (3)

R.H.S : $z_1z_3 = 3(5+4i) = 15 + 12i$

$z_2z_3 = -7i(5+4i) = -35i - 28i^2 = 28 - 35i$

$z_1z_3 + z_2z_3 = 15 + 12i + 28 - 35i$

$= 43 - 23i$ - (4)

(3) = (4) $\mathbf{(z_1 + z_2)z_3 = z_1z_3 + z_2z_3}$

---

(3) Find the additive & multiplicative inverse of following complex numbers.

$\mathbf{z = a + ib} \qquad \mathbf{z^{-1} = \frac{a}{a^2+b^2} + i\frac{-b}{a^2+b^2}}$

(i) $\mathbf{z_1 = 2 + 5i}$ a = 2 & b = 5

additive inverse $-z_1 = -2 - 5i$

multiplicative inverse

$z_1^{-1} = \frac{2}{2^2+5^2} + i\frac{-5}{2^2+5^2} = \mathbf{\frac{2}{29} - \frac{5i}{29}}$

(ii) $\mathbf{z_2 = -3 - 4i}$ a = -3 & b = -4

additive inverse: $-z_2 = -(-3-4i) = 3 + 4i$

Multiplicative inverse :

$z_1^{-1} = \frac{-3}{(-3)^2+(-4)^2} + i\frac{-(-4)}{(-3)^2+(-4)^2} = \mathbf{\frac{-3}{25} + \frac{4\,i}{25}}$

(ii) $\mathbf{z_3 = 1 + i}$ a = 1 & b = 1

$-z_3 = -(1+i) = -1 - i$

$z_3^{-1} = \frac{1}{1^2+1^2} + i\frac{(-1)}{1^2+1^2} = \mathbf{\frac{1}{2} - \frac{i}{2}}$

EXERCISE 2.4

4. u = ? $\mathbf{v = 3 - 4i} \quad \mathbf{w = 4 + 3i}$ & $\mathbf{\frac{1}{u} = \frac{1}{v} + \frac{1}{w}}$

$\frac{1}{v} = \frac{1}{3-4i} \times \frac{3+4i}{3+4i} = \frac{3+4i}{3^2+4^2} = \frac{3+4i}{25}$

$\frac{1}{w} = \frac{1}{4+3i} \times \frac{4-3i}{4-3i} = \frac{4-3i}{4^2+3^2} = \frac{4-3i}{25}$

$\frac{1}{u} = \frac{1}{v} + \frac{1}{w} = \frac{3+4i}{25} + \frac{4-3i}{25} = \frac{7+i}{25}$

$\frac{1}{u} = \frac{7+i}{25}$

$u = \frac{1}{\frac{7+i}{25}} = \frac{25}{7+i} \times \frac{7-i}{7-i} = \frac{25(7-i)}{7^2+1^2}$

$= \frac{25(7-i)}{50} = \frac{1}{2}(7 - i)$

---

5. $\mathbf{z = \bar{z}} \qquad \mathbf{z = a + ib}$

$\Leftrightarrow x + iy = \overline{x + iy} \qquad \bar{z} = a - ib$

$\Leftrightarrow x + iy = x - iy \qquad z + \bar{z} = 2a$

$\Leftrightarrow 2iy = x - x \qquad z + \bar{z} = 2Re(z)$

$\Leftrightarrow 2iy = 0 \qquad Re\,(z) = \frac{z+z}{2}$

$\Leftrightarrow y = 0 \qquad z - \bar{z} = 2ib$

$\Leftrightarrow z = x \qquad z - \bar{z} = 2i\ im(z)$

$\Leftrightarrow$ **z is purely real** $\qquad im\,(z) = \frac{z-\bar{z}}{2i}$

---

6. Find the least tive integer n such that

$\mathbf{(\sqrt{3} + i)^n}$ **(i) real (i) Imaginary**

(i) $(\sqrt{3} + i)^1 = \sqrt{3} + i$ Complex no.

(ii) $(\sqrt{3} + i)^2 = (\sqrt{3})^2 + 2\sqrt{3}i + i^2$

$= 3 + 2\sqrt{3}i - 1 = 2 + 2\sqrt{3}i$

(iii) $(\sqrt{3} + i)^3 = (\sqrt{3} + i)^2(\sqrt{3} + i)$

$= (2 + 2\sqrt{3}i)(\sqrt{3} + i)$

$= 2\sqrt{3} + 2i + 6i + 2\sqrt{3}i^2$

$= 2\sqrt{3} + 8i - 2\sqrt{3}$

$= 8i$ purely imaginary

(iv) $(\sqrt{3} + i)^4 = (\sqrt{3} + i)^3(\sqrt{3} + i) = 8i(\sqrt{3} + i)$

$= 8\sqrt{3}i + 8i^2 = -8 + 8\sqrt{3}i$

(v) $(\sqrt{3} + i)^5 = (\sqrt{3} + i)^3(\sqrt{3} + i)^2$

$= 8i(2 + 2\sqrt{3}i) = 16i + 16\sqrt{3}i^2$

(vi) $(\sqrt{3} + i)^6 = (\sqrt{3} + i)^3(\sqrt{3} + i)^3 = 8i(8i) = 64i^2$

$= -64$ purely real

$n = 6 \quad (\sqrt{3} + i)^n$ is real

$n = 3 \quad (\sqrt{3} + i)^n$ is imaginary

7. (i) Show that $(2+i\sqrt{3})^{10}-(2-i\sqrt{3})^{10}$ purely imaginary

Solution :

$z=(2+i\sqrt{3})^{10}-(2-i\sqrt{3})^{10}$

$\bar{z}=\overline{(2+i\sqrt{3})^{10}-(2-i\sqrt{3})^{10}}$

$=\overline{(2+i\sqrt{3})^{10}}-\overline{(2-i\sqrt{3})^{10}}$

$=\left(\overline{2+i\sqrt{3}}\right)^{10}-\left(\overline{2-i\sqrt{3}}\right)^{10}$

$=\left(2-i\sqrt{3}\right)^{10}-\left(2+i\sqrt{3}\right)^{10}$

$=-\left[\left(2+i\sqrt{3}\right)^{10}-\left(2-i\sqrt{3}\right)^{10}\right]$

$\bar{z}=-z$ ∴ **z is purely imaginary .**

EXERCISE 2.5

(2) For any two complex numbers z_1 and z_2 such that $|z_1|=|z_2|=1$ $z_1z_2\neq-1$ then show that $\frac{z_1+z_2}{1+z_1z_2}$ is a real number.

Solution :

$|z_1|=1$ $|z_2|=1$

$|z_1|^2=1$ $|z_2|^2=1$

$z_1\bar{z}_1=1$ $z_2\bar{z}_2=1$

$z_1=\frac{1}{\bar{z}_1}$ $z_2=\frac{1}{\bar{z}_2}$

let $z=\frac{z_1+z_2}{1+z_1z_2}=\frac{\frac{1}{\bar{z}_1}+\frac{1}{\bar{z}_2}}{1+\frac{1}{\bar{z}_1}\cdot\frac{1}{\bar{z}_2}}=\frac{\frac{\bar{z}_1+\bar{z}_2}{\bar{z}_1\bar{z}_2}}{\frac{\bar{z}_1\bar{z}_2+1}{\bar{z}_1\bar{z}_2}}=\frac{\bar{z}_1+\bar{z}_2}{1+\bar{z}_1\bar{z}_2}=\frac{\overline{z_1+z_2}}{\overline{1+z_1z_2}}$

$=\overline{\left(\frac{z+z_2}{1+z_1z_2}\right)}$

$z=\bar{z}$ therefore z is purely real .

---

3. Which one of the point $10-8i$, $11+6i$ is closest to $1+i$.

Solution : A , B , C rep c'x numbers

$z_1=10-8i$, $z_2=11+6i$, $z_3=1+i$

$AC=|z_1-z_3|=|10-8i-1-i|=|9-9i|$

$=\sqrt{9^2+(-9)^2}=\sqrt{81+81}=\sqrt{162}$

$BC=|z_2-z_3|=|11+6i-1-i|=|10+5i|$

$=\sqrt{10^2+5^2}=\sqrt{100+25}=\sqrt{125}$

$\sqrt{125}<\sqrt{162}$ ∴ **$11+6i$ is closer to $1+i$**

---

(4) If $|z|=3$, Show $7\leqslant|z+6-8i|<13$.

Let $z_1=6-8i$

$|z_1|=\sqrt{6^2+(-8)^2}=\sqrt{36+64}=\sqrt{100}=10$

Wehave $||z|-|z_1||\leq|z+z_1|\leq|z|+|z_1|$

$|3-10|\leq|z+6-8i|\leq3+10.$

$|-7|\leq|z+6-8i|\leq13$

$\Rightarrow 7\leq|z+6-8i|\leq13.$

---

(5) If $|z|=1$ show that $2\leq|z^2-3|<4$

Solution :

let $z_1=-3$ ∴ $|z_1|=|-3|=3$.

$|z|=1$ $\Rightarrow|z^2|=|z|^2=1^2=1.$

we know $||z|^2-|z_1||\leq|z^2+(-3)|\leq|z|^2+|z_1|$

$|1^2-3|\leq|z^2-3|\leq1+3$

$|-2|\leq|z^2-3|\leq4$

$2\leq|z^2-3|\leq4$

---

(8) The area of triangle formed by the vertices z , iz , z + iz is 50 sq.units. find |z| .

Solution :

Let A , B , C represent C'x nos z , iz , z + iz respectively.

$AB=|z-iz|=|z(1-i)|=|z||1-i|$

$=|z|\sqrt{1^2+(-1)^2}=|z|\sqrt{1+1}=\sqrt{2}|z|$

$BC=|iz-z-iz|=|-z|=|z|$

$AC=|z-(z+iz)|=|z-z-iz|=|-iz|$

$=|-i||z|=|z|$

AC = BC isosceles right triangle.

$AC^2+BC^2=|z|^2+|z|^2=2|z|^2=AB^2$

∴ ABC is an isosceles right triangle .

Area $=\frac{1}{2}BC\,AC=50$

$|z||z|=100$

$|z|^2=100$

$|z|=10$ $|z|=-10$ not possible

---

(9) S.T $z^3+2\bar{z}=0$ has five solution .

Solution :

$z^3+2\bar{z}=0$ - (1)

$z^3=-2\bar{z}$

$|z|^3=|-2||\bar{z}|$

$|z|^3=2|z|$

$|z|^3-2|z|=0$

$|z|(|z|^2-2)=0$

$|z|=0$ $|z|^2-2=0$

$z=0$ $|z|^2=2\Rightarrow z\bar{z}=2$

$\bar{z}=\frac{2}{z}$

sub in (1) $z^3+2\cdot\frac{2}{z}=0\Rightarrow z^4+4=0$

$|z|=0$ $z^4+4=0$

$\Rightarrow z=0$ $z^4+4=0$ gives 4 solution

∴ It has five solution.

$z = a + ib$.
$|z| = \sqrt{a^2+b^2}$
$\sqrt{a+ib} = \pm\left(\sqrt{\frac{|z|+a}{2}} + i\frac{b}{|b|}\sqrt{\frac{|z|-a}{2}}\right)$

10 (i) Find the square root of $4+3i$

$z = 4+3i \quad a = 4 \quad b = 3$

$|z| = \sqrt{4^2+3^2} = \sqrt{16+9} = \sqrt{25} = 5$

$\sqrt{a+ib} = \pm\left(\sqrt{\frac{|z|+a}{2}} + \frac{ib}{|b|}\sqrt{\frac{|z|-a}{2}}\right)$

$\sqrt{4+3i} = \pm\left[\sqrt{\frac{5+4}{2}} + i\frac{3}{|3|}\sqrt{\frac{5-4}{2}}\right] = \pm\left(\frac{\sqrt{9}}{\sqrt{2}} + i\frac{3}{3}\frac{\sqrt{1}}{\sqrt{2}}\right)$

$= \pm\left(\frac{\sqrt{9}}{\sqrt{2}} + \frac{\sqrt{1}}{\sqrt{2}}i\right) = \pm\left(\frac{3}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)$

(ii) Find the square root of $-6+8i$

$z = -6+8i \quad a = -6 \quad b = 8$

$|z| = \sqrt{(-6)^2+8^2} = \sqrt{36+64} = \sqrt{100} = 10$

$\sqrt{a+ib} = \pm\left(\sqrt{\frac{|z|+a}{2}} + \frac{ib}{|b|}\sqrt{\frac{|z|-a}{2}}\right)$

$\sqrt{-6+8i} = \pm\left(\sqrt{\frac{10+(-6)}{2}} + i\frac{8}{|8|}\sqrt{\frac{10-(-6)}{2}}\right)$

$= \pm\left(\sqrt{\frac{10-6}{2}} + i\frac{8}{8}\sqrt{\frac{10+6}{2}}\right) = \pm\left(\sqrt{\frac{4}{2}} + i\sqrt{\frac{16}{2}}\right)$

$= \pm\left(\sqrt{2} + i\sqrt{8}\right) = \pm\left(\sqrt{2} + i2\sqrt{2}\right)$

(iii) Find the square root of $-5-12i$

$z = -5-12i \quad a = -5 \quad b = -12$

$|z| = \sqrt{(-5)^2+(-12)^2} = \sqrt{25+144} = \sqrt{169} = 13$

$\sqrt{a+ib} = \pm\left(\sqrt{\frac{|z|+a}{2}} + i\frac{b}{|b|}\sqrt{\frac{|z|-a}{2}}\right)$

$= \pm\left(\sqrt{\frac{13+(-5)}{2}} + i\left(\frac{-12}{12}\right)\sqrt{\frac{13-(-5)}{2}}\right) = \pm\left(\sqrt{\frac{8}{2}} + \left(\frac{-12}{12}\right)i\sqrt{\frac{18}{2}}\right)$

$= \pm\left(\sqrt{4} - i\sqrt{9}\right) = \pm(2-i3) = \pm(2-3i)$

Note :

$\mathbf{z = a+ib}$

$\mathbf{|z| = \sqrt{a^2+b^2}}$

$\mathbf{(a+b)^2 = a^2+2ab+b^2}$

$\mathbf{(a-b)^2 = a^2-2ab+b^2}$

EXERCISE 2.6

(1) $z = x+iy$ is a complex number such that $\left|\frac{z-4i}{z+4i}\right| = 1$. Show that locus of z is real axis.

Solution : $\quad z = x+iy$

Given: $\left|\frac{z-4i}{z+4i}\right| = 1$

$\frac{|z-4i|}{|z+4i|} = 1$
$|z-4i| = |z+4i|$
$|x+iy-4i| = |x+iy+4i|$
$|x+i(y-4)| = |x+i(y+4)|$
$\sqrt{x^2+(y-4)^2} = \sqrt{x^2+(y+4)^2}$
$x^2+(y-4)^2 = x^2+(y+4)^2$
$x^2+y^2-8y+16 = x^2+y^2+8y+16$
$\Rightarrow -16y = 0$
$\mathbf{y = 0}$ **equation of x – axis**

EXERCISE 2.7

1. (iv) $\dfrac{i-1}{\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}}$

Consider $i-1 = -1+i = r(\cos\theta + i\sin\theta)$

$a = -1 \quad b = 1$

$r = \sqrt{a^2+b^2} = \sqrt{(-1)^2+1^2} = \sqrt{1+1} = \sqrt{2}$

$\alpha = \tan^{-1}\left|\frac{b}{a}\right| = \tan^{-1}\left|\frac{1}{-1}\right| = \tan^{-1}(1) = \frac{\pi}{4}$

$\theta = \pi - \alpha = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$ ($-1+i$ lies in II quadrant)

$i-1 = -1+i = \sqrt{2}\left[\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right]$

$\frac{i-1}{\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}} = \frac{\sqrt{2}\left[\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right]}{\cos\frac{\pi}{3}+i\sin\frac{3\pi}{4}}$

$= \sqrt{2}\left[\cos\left(\frac{3\pi}{4}-\frac{\pi}{3}\right) + i\sin\left(\frac{3\pi}{4}-\frac{\pi}{3}\right)\right]$

$= \sqrt{2}\left[\cos\frac{9\pi-4\pi}{12} + i\sin\frac{9\pi-4\pi}{12}\right]$

$= \sqrt{2}\left[\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right]$

$\mathbf{\frac{i-1}{\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}} = \sqrt{2}\left[\cos\left(2k\pi+\frac{5\pi}{12}\right) + i\sin\left(2k\pi+\frac{5\pi}{12}\right)\right]}$

(2) Find the rectangular form.

(i) $\mathbf{\left[\cos\left(\frac{\pi}{6}\right) + i\sin\frac{\pi}{6}\right]\left[\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right]}$

$= \cos\left(\frac{\pi}{6}+\frac{\pi}{12}\right) + i\sin\left(\frac{\pi}{6}+\frac{\pi}{12}\right)$

$= \cos\left(\frac{2\pi+\pi}{12}\right) + i\sin\left(\frac{2\pi+\pi}{12}\right) = \cos\left(\frac{3\pi}{12}\right) + i\sin\left(\frac{3\pi}{12}\right)$

$\mathbf{= \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + i\cdot\frac{1}{\sqrt{2}}}$

(ii) $\frac{\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}}{2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)} = \frac{\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)}{2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)}$

$= \frac{1}{2}\left[\cos\left(-\frac{\pi}{6}-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{6}-\frac{\pi}{3}\right)\right]$

$= \frac{1}{2}\left[\cos\left(\frac{-\pi-2\pi}{6}\right) + i\sin\left(\frac{-\pi-2\pi}{6}\right)\right]$

$= \frac{1}{2}\left[\cos\left(-\frac{3\pi}{6}\right) + i\sin\left(-\frac{3\pi}{6}\right)\right] = \frac{1}{2}\left[\cos\left(\frac{-\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right]$

$= \frac{1}{2}\left[\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right] = \frac{1}{2}(0-i) \quad = \mathbf{\frac{1}{2}i}$

---

(3) If $(x_1+iy_1)(x_2+iy_2)\dots(x_n+iy_n) = a+ib$

show that $(x_1^2+y_1^2)(x_2^2+y_2^2)\dots(x_n^2+y_n^2) = a^2+b^2$

$\sum_{r=1}^{n}\tan^{-1}\left(\frac{y_r}{x_r}\right) = k\pi + \tan^{-1}\left(\frac{b}{a}\right) \qquad k \in z$

Solution :

(i) $(x_1+iy_1)(x_2+iy_2)\dots(x_n+iy_n) = a+ib$

$|(x_1+iy_1)(x_2+iy_2)\dots(x_n+iy_n)| = |a+ib|$

$|(x_1+iy_1)||(x_2+iy_2)|\dots|(x_n+iy_n)| = |a+ib|$

$\sqrt{x_1^2+y_1^2}\sqrt{x_2^2+y_2^2}\cdots\sqrt{x_n^2+y_n^2} = \sqrt{a^2+b^2}$

On squaring

$(x_1^2+y_1^2)(x_2^2+y_2^2)\dots(x_n^2+y_n^2) = a^2+b^2$

(ii) $\arg[(x_1+iy_1)(x_2+iy_2)\dots(x_n+iy_n)] = \arg(a+ib)$

$\arg(x_1+iy_1) + \arg(x_2+iy_2) + \cdots + \arg(x_n+iy_n)$

$= \arg(a+ib)$

$\Rightarrow \tan^{-1}\left(\frac{y_1}{x_1}\right) + \tan^{-1}\left(\frac{y_2}{x_2}\right) + \cdots + \tan^{-1}\left(\frac{y_n}{x_n}\right) = \tan^{-1}\left(\frac{b}{a}\right)$

∴ Given solution

$\tan^{-1}\left(\frac{y_1}{x_1}\right) + \tan^{-1}\left(\frac{y_2}{x_2}\right) + \cdots + \tan^{-1}\left(\frac{y_n}{x_n}\right)$

$= k\pi + \tan^{-1}\left(\frac{b}{a}\right)$

EXERCISE 2.8

2) Show that $\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^5 + \left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)^5 = -\sqrt{3}$

Solution : $\frac{\sqrt{3}}{2}+\frac{i}{2} = r(\cos\theta + i\sin\theta)$

$r = \sqrt{a^2+b^2} = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4}+\frac{1}{4}} = 1$

$\alpha = \tan^{-1}\left|\frac{b}{a}\right| = \tan^{-1}\left|\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right| = \tan^{-1}\left|\frac{1}{\sqrt{3}}\right| = \frac{\pi}{6}$

$\theta = \alpha = \frac{\pi}{6} \qquad \because \frac{\sqrt{3}}{2}+\frac{i}{2}$ lies in I Quad

$\frac{\sqrt{3}}{2}+\frac{i}{2} = 1\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

$\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^5 = \left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^5 = \mathbf{\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}}$

Similarly $\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)^5 = \cos\frac{5\pi}{6} - i\sin\frac{5\pi}{6}$

$\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^5 + \left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)^5 = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6} + \cos\frac{5\pi}{6} - i\sin\frac{5\pi}{6}$

$= \mathbf{2\cos\frac{5\pi}{6} = 2\cos 150^\circ}$

$= \mathbf{2\cos(180^\circ - 30^\circ) = 2[-\cos 30^\circ] = -2\cdot\frac{\sqrt{3}}{2} = -\sqrt{3}}$

5. Solve: $z^3 + 27 = 0$

$\mathbf{z^3 = -27 \quad = 27 \times -1}$

$z = (27)^{1/3}(-1)^{1/3}$

$z = (27)^{\frac{1}{3}}[\cos\pi + i\sin\pi]^{\frac{1}{3}}$

$z = 3[\cos(2k\pi+\pi) + i\sin(2k\pi+\pi)]^{\frac{1}{3}}$

$= 3\left[\cos(2k+1)\frac{\pi}{3} + i\sin(2k+1)\frac{\pi}{3}\right] \quad K = 0, 1, 2$

$= 3\,\text{cis}\,(2k+1)\frac{\pi}{3}$

$\mathbf{k = 0;\quad z_1 = 3\,cis\,\frac{\pi}{3}}$

$\mathbf{k = 1;\quad z_2 = 3\,cis\,\frac{3\pi}{3} = 3\,cis\pi = -3}$

$\mathbf{k = 2;\quad z_3 = 3\,cis\,\frac{5\pi}{3}}$

---

(5) $\omega \neq 1$ cube roots of unity. S.T. roots of eqn

$(z-1)^3 + 8 = 0$ are $-1, +1-2\omega, 1-2\omega^2$

Solution :

$(z-1)^3 + 8 = 0$

$(z-1)^3 = -8$

$(1-z)^3 = 8 = 2^3$

$\left(\frac{1-z}{2}\right)^3 = 1$

$\frac{1-z}{2} = (1)^{1/3}$

$\frac{1-z}{2} = 1 \quad \frac{1-z}{2} = \omega \quad \frac{1-z}{2} = \omega^2$

$1-z = 2 \quad 1-z = 2\omega \quad 1-z = 2\omega^2$

$\Rightarrow z = -1 \quad z = 1-2\omega \quad z = 1-2\omega^2$

$\therefore$ **roots are $-1, 1-2\omega, 1-2\omega^2$**

---

(7) Find he value $\sum_{k=1}^{8}\left(\cos\frac{2k\pi}{9} + i\sin\frac{2k\pi}{9}\right)$

Solution :

Let $x = \cos\frac{2k\pi}{9} + i\sin\frac{2k\pi}{9}$

$k = 1 \quad x_1 = \cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9} = \omega$

$k = 2 \quad x_2 = \cos\frac{4\pi}{9} + i\sin\frac{4\pi}{9} = \left(\cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9}\right)^2 = \omega^2$

$k = 3 \quad x_3 = \cos\frac{6\pi}{9} + i\sin\frac{6\pi}{9} = \left(\cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9}\right)^3 = \omega^3$

$k = 4 \quad x_4 = \cos\frac{8\pi}{9} + i\sin\frac{8\pi}{9} = \left(\cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9}\right)^4 = \omega^4$

$k = 5 \quad x_5 = \cos\frac{10\pi}{9} + i\sin\frac{10\pi}{9} = \left(\cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9}\right)^5 = \omega^5$

$k = 6 \quad x_6 = \cos\frac{12\pi}{9} + i\sin\frac{12\pi}{9} = \left(\cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9}\right)^6 = \omega^6$

$k = 7 \quad x_7 = \cos\frac{14\pi}{9} + i\sin\frac{14\pi}{9} = \left(\cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9}\right)^7 = \omega^7$

$k = 8 \quad x_8 = \cos\frac{16\pi}{4} + i\sin\frac{16\pi}{9} = \left(\cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9}\right)^8 = \omega^8$

$\mathbf{\sum_{k=1}^{8}\cos\frac{2k\pi}{9} + i\sin\frac{2k\pi}{9}}$

$\mathbf{= \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 + \omega^7 + \omega^8 = -1}$

$\mathbf{(\because 1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 + \omega^7 + \omega^8 = 0)}$

(9) If $\mathbf{z = 2 - 2i}$. Find the rotation of $\mathbf{z}$ by $\boldsymbol{\theta}$ radians by counter clockwise direction.

$\mathbf{z = 2 - 2i = r(\cos\theta + i\sin\theta)}$

$a = 2 \quad b = -2$

$r = \sqrt{a^2 + b^2} = \sqrt{2^2 + (-2)^2} = \sqrt{4 + 4} = \sqrt{8} = 2\sqrt{2}$

$\alpha = \tan^{-1}\left|\frac{b}{a}\right| = \tan^{-1}\left|\frac{-2}{2}\right| = \tan^{-1}(1)$

$\alpha = \frac{\pi}{4} \Rightarrow \theta = -\alpha = -\frac{\pi}{4}$ lies in IV Quadrant

$\therefore z = 2 - 2i = 2\sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right] = 2\sqrt{2}e^{-i\frac{\pi}{4}}$

(i) rotated by $\frac{\pi}{3}$

$\mathbf{z_1 = 2\sqrt{2}e^{-i\frac{\pi}{4}} \cdot e^{i\frac{\pi}{3}} = 2\sqrt{2}e^{i(-\frac{\pi}{4}+\frac{\pi}{3})} = 2\sqrt{2}e^{i\frac{\pi}{12}}}$

(ii) rotated by $\frac{2\pi}{3}$

$\mathbf{z_2 = 2\sqrt{2}e^{-i\frac{\pi}{4}} \cdot e^{i\frac{2\pi}{3}} = 2\sqrt{2}e^{i(-\frac{\pi}{4}+\frac{2\pi}{3})} = 2\sqrt{2}e^{i\frac{5\pi}{12}}}$

(iii) rotated by $\frac{3\pi}{2}$

$\mathbf{z_3 = 2\sqrt{2}e^{-i\frac{\pi}{4}} \cdot e^{i\frac{3\pi}{2}} = 2\sqrt{2}e^{i(-\frac{\pi}{4}+\frac{3\pi}{2})} = 2\sqrt{2}e^{i\frac{5\pi}{4}}}$

---

(8) $\neq \mathbf{1}$, S.T.

(i) $\mathbf{(1 - \omega + \omega^2)^6 + (1 + \omega - \omega^2)^6 = 128}$

$L \cdot H \cdot S = (1 - \omega + \omega^2)^6 + (1 + \omega - \omega^2)^6$

$= (1 + \omega^2 - \omega)^6 + (1 + \omega - \omega^2)^6$

$= (-\omega - \omega)^6 + (-\omega^2 - \omega^2)^6 = (-2\omega)^6 + (-2\omega^2)^6$

$= (-2)^6\omega^6 + (-2)^6(\omega^2)^6 = 64\omega^6 + 64\omega^{12}$

$= 64 + 64 = \mathbf{128}$

(ii) $\mathbf{(1 + \omega)(1 + \omega^2)(1 + \omega^4)(1 + \omega^8)\ldots\left(1 + \omega^{2^{11}}\right) = 1}$

L.H.S

$(1 + \omega)(1 + \omega^2)(1 + \omega^4)(1 + \omega^8)\ldots\left(1 + \omega^{2^{11}}\right)$

$= (1 + \omega)(1 + \omega^2)(1 + \omega^4)(1 + \omega^8)(1 + \omega^{16})$

$(1 + \omega^{32})(1 + \omega^{64})(1 + \omega^{128})\ldots\left(1 + \omega^{2^{11}}\right)$

$= (1 + \omega)(1 + \omega^2)(1 + \omega)(1 + \omega^2)(1 + \omega)(1 + \omega^2)(1 + \omega)$

$(1 + \omega^2) \quad (1 + \omega)(1 + \omega^2)(1 + \omega)(1 + \omega^2)$

$= [(1 + \omega)(1 + \omega^2)]^6 = [1 + \omega^2 + \omega + \omega^3]^6$

$= (0 + 1)^6 = 1^6 = \mathbf{1}$

---

(3) Find the value of $\left[\frac{1+\sin\frac{\pi}{10}+i\cos\frac{\pi}{10}}{1+\sin\frac{\pi}{10}-i\cos\frac{\pi}{10}}\right]$

<u>Solution :</u>

let $\mathbf{z = \sin\frac{\pi}{10} + i\cos\frac{\pi}{10}}$

$\because |z| = 1 \Rightarrow z^{-1} = \bar{z} = \sin\frac{\pi}{10} - i\cos\frac{\pi}{10}$

$\therefore \left[\frac{1+\sin\frac{\pi}{10}+i\cos\frac{\pi}{10}}{1+\sin\frac{\pi}{10}-i\cos\frac{\pi}{10}}\right]^{10} = \left[\frac{1+z}{1+\frac{1}{z}}\right]^{10} = \left[\frac{1+z}{\frac{z+1}{z}}\right]^{10} = (z)^{10}$

$= \left[\sin\frac{\pi}{10} + i\cos\frac{\pi}{10}\right]^{10}$

$= i^{10}\left[\cos\frac{\pi}{10} - i\sin\frac{\pi}{10}\right]^{10}$

$= i^8 \cdot i^2\left[\cos\frac{\pi}{10} \times 10 - i\sin\frac{\pi}{10} \times 10\right]$

$\mathbf{= -1[\cos\pi - i\sin\pi] = -1(-1) = 1}$

<u>5 MARKS</u>

<u>EXAMPLE 2.8(ii)</u>

PROVE : $\left(\frac{19+9i}{5-3i}\right)^{15} - \left(\frac{8+i}{1+2i}\right)^{15}$ is purely imaginary

<u>Solution :</u>

$\frac{19+9i}{5-3i} = \frac{19+9i}{5-3i} \times \frac{5+3i}{5+3i} = \frac{95+57i+45i+27i^2}{5^2+3^2} = \frac{95+102i-27}{25+9}$

$= \frac{68+102i}{34} = 34\frac{(2+3i)}{34} = 2 + 3i$

$\frac{8+i}{1+2i} = \frac{8+i}{1+2i} \times \frac{1-2i}{1-2i} = \frac{8-16i+i-2i^2}{1^2+2^2}$

$= \frac{8-15i+2}{1+4} = \frac{10-15i}{5} = \frac{5(2-3i)}{5} = 2 - 3i$

Let $z = \left(\frac{19+9i}{5-3i}\right)^{15} - \left(\frac{8+i}{1+2i}\right)^{15} = (2 + 3i)^{15} - (2 - 3i)^{15}$

$\bar{z} = \overline{(2 + 3i)^{15} - (2 - 3i)^{15}}$

$= \overline{(2 + 3i)^{15}} - \overline{(2 - 3i)^{15}}$

$= (\overline{2 + 3i})^{15} - (\overline{2 - 3i})^{15}$

$= (2 - 3i)^{15} - (2 + 3i)^{15}$

$= -\left[(2 + 3i)^{15} - (2 - 3i)^{15}\right] = -z$

$\Rightarrow$ $\mathbf{z = -\bar{z}}$ $\therefore$ **z is purely imaginary**

<u>EXERCISE 2.4 7 (ii)</u>

PROVE $\left(\frac{19-7i}{9+i}\right)^{12} + \left(\frac{20-5i}{7-6i}\right)^{12}$ is real .

<u>Solution :</u>

$\frac{19-7i}{9+i} = \frac{19-7i}{9+i} \times \frac{9-i}{9-i} = \frac{171-19i-63i+7i^2}{9^2+1^2}$

$= \frac{171-82i-7}{81+1} = \frac{164-82i}{82}$

$= \frac{82(2-i)}{82} = 2 - i$

$\frac{20-5i}{7-6i} = \frac{20-5i}{7-6i} \times \frac{7+6i}{7+6i}$

$= \frac{140+120i-35i-30i^2}{7^2+6^2}$

$= \frac{140+85i+30}{49+36}$

$= \frac{170+85i}{85} = \frac{85(2+i)}{85} = 2 + i$

Let $z = \left(\frac{19-7i}{9+i}\right)^{12} + \left(\frac{20-5i}{7-6i}\right)^{12}$

$z = (2 - i)^{12} + (2 + i)^{12}$

$\bar{z} = \overline{(2 - i)^{12} + (2 + i)^{12}}$

$= \overline{(2 - i)^{12}} + \overline{(2 + i)^{12}}$

$= (\overline{2 - i})^{12} + (\overline{2 + i})^{12}$

$= (2 + i)^{12} + (2 - i)^{12} = z$

$\therefore$ $\mathbf{\bar{z} = z}$, **z is real .**

EXAMPLE 2.14

show that the points $\mathbf{1}$, $\frac{-1}{2}+i\frac{\sqrt{3}}{2}$ and $\frac{-1}{2}-i\frac{\sqrt{3}}{2}$ forms a equilateral triangle.

Solution :

Let A, B, C represent $z_1 = 1$; $z_2 = \frac{-1}{2}+i\frac{\sqrt{3}}{2}$; $z_3 = \frac{-1}{2}-i\frac{\sqrt{3}}{2}$

$$AB = |z_1 - z_2| = \left|1-\left(\frac{-1}{2}+i\frac{\sqrt{3}}{2}\right)\right| = \left|1+\frac{1}{2}-i\frac{\sqrt{3}}{2}\right|$$

$$= \left|\frac{3}{2}-i\frac{\sqrt{3}}{2}\right| = \sqrt{\left(\frac{3}{2}\right)^2+\left(-\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{9}{4}+\frac{3}{4}} = \sqrt{\frac{12}{4}} = \sqrt{3}$$

$$BC = |z_2 - z_3| = \left|\frac{-1}{2}+i\frac{\sqrt{3}}{2}-\left(\frac{-1}{2}-i\frac{\sqrt{3}}{2}\right)\right|$$

$$= \left|\frac{-1}{2}+i\frac{\sqrt{3}}{2}+\frac{1}{2}+i\frac{\sqrt{3}}{2}\right| = \left|i\frac{2\sqrt{3}}{2}\right| = |i\sqrt{3}|$$

$$= \sqrt{0^2+\sqrt{3}^2} = \sqrt{3}$$

$$AC = |z_1 - z_3| = \left|1-\left(\frac{-1}{2}-i\frac{\sqrt{3}}{2}\right)\right| = \left|1+\frac{1}{2}+i\frac{\sqrt{3}}{2}\right|$$

$$= \left|\frac{3}{2}+i\frac{\sqrt{3}}{2}\right| = \sqrt{\left(\frac{3}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{9}{4}+\frac{3}{4}} = \sqrt{\frac{12}{4}} = \sqrt{3}$$

$\mathbf{AB = BC = AC}$ **. Therefore It forms equilateral triangle .**

EXAMPLE 2.15

$|z_1| = |z_2| = |z_3| = r$, $z_1+z_2+z_3 \neq 0$; S.T $\left|\frac{z_1z_2+z_2z_3+z_3z_1}{z_1+z_2+z_3}\right| = r$

Solution: $|z_1| = r \Rightarrow |z_1|^2 = r^2 \Rightarrow z_1\overline{z_1} = r^2 \Rightarrow z_1 = \frac{r^2}{\overline{z_1}}$,

similarly $z_2 = \frac{r^2}{\overline{z_2}}$, $z_3 = \frac{r^2}{\overline{z_3}}$

$$z_1+z_2+z_3 = \frac{r^2}{\bar{z}_1}+\frac{r^2}{\bar{z}_2}+\frac{r^2}{\bar{z}_3} = r^2\left(\frac{1}{\bar{z}_1}+\frac{1}{\bar{z}_2}+\frac{1}{\bar{z}_3}\right)$$

$$= r^2\left(\frac{\bar{z}_2\bar{z}_3+\bar{z}_1\bar{z}_3+\bar{z}_1\bar{z}_2}{\bar{z}_1\bar{z}_1\bar{z}_3}\right) = r^2\frac{(\overline{z_2z_3}+\overline{z_1z_3}+\overline{z_1z_2})}{\overline{z_1z_2z_3}}$$

$$|z_1+z_2+z_3| = \frac{|r^2(\overline{z_2z_1+z_1z_3+z_1z_2})|}{|\overline{z_1z_2z_3}|} = |r^2|\frac{|z_2z_3+z_1z_3+z_1z_2|}{|z_1||z_2||z_3|}$$

$$|z_1+z_2+z_3| = r^2\frac{|z_2z_3+z_1z_3+z_1z_2|}{r.r.r}$$

$$\frac{r^3}{r^2} = \frac{|z_1z_2+z_2z_3+z_3z_1|}{|z_1+z_2+z_3|} \Rightarrow r = \frac{|z_1z_2+z_2z_3+z_3z_1|}{|z_1+z_2+z_3|}$$

---

(7) If z_1 , z_2 and z_3 are 3 complex nos, such that

$\mathbf{|z_1| = 1 , |z_2| = 2 , |z_3| = 3 , |z_1+z_2+z_3| = 1}$

Show that $|9z_1z_2+4z_1z_3+z_2z_3| = 6$.

Solution : $|z_1| = 1$ $\quad |z_2| = 2$ $\quad |z_3| = 3$

$|z_1|^2 = 1$ $\quad |z_2|^2 = 4$ $\quad |z_3|^2 = 9$

$z_1\overline{z_1} = 1$ $\quad z_2\overline{z_2} = 4$ $\quad z_3\overline{z_3} = 9$

$z_1 = \frac{1}{\overline{z_1}}$ $\quad z_2 = \frac{4}{\overline{z_2}}$ $\quad z_3 = \frac{9}{\overline{z_3}}$

$$|z_1+z_2+z_3| = 1 \Rightarrow \left|\frac{1}{\overline{z_1}}+\frac{4}{\overline{z_2}}+\frac{9}{\overline{z_3}}\right| = 1$$

$$\left|\frac{\bar{z}_2\bar{z}_3+4\bar{z}_1\bar{z}_3+9\bar{z}_1\bar{z}_2}{\bar{z}_1\bar{z}_2\overline{z_3}}\right| = 1$$

$$\frac{|\overline{z_2z_3}+4\overline{z_1z_3}+9\overline{z_1z_2}|}{|\overline{z_1z_2z_3}|} = 1$$

$$|\overline{z_2z_3+4z_1z_3+9z_1z_2}| = |z_1z_2z_3|$$

$$|z_2z_3+4z_1z_3+9z_1z_2| = |z_1|\,|z_2||z_3|$$

$$\Rightarrow |z_2z_3+4z_1z_3+9z_1z_2| = 6$$

Exercise 2.5 (9): S.T $\mathbf{z^3+2\bar{z} = 0}$ **has five solution .**

Solution :

$z^3+2\bar{z} = 0$ $\qquad$ - (1)

$z^3 = -2\bar{z}$

$|z|^3 = |-2||\bar{z}| \Rightarrow |z|^3 = 2|z|$

$|z|^3 - 2|z| = 0 \Rightarrow |z|(|z|^2-2) = 0$

$|z| = 0$ $\quad$ & $\quad$ $|z|^2 - 2 = 0$

$z = 0$ $\qquad$ $|z|^2 = 2 \Rightarrow z\bar{z} = 2 \Rightarrow \bar{z} = \frac{2}{z}$

sub in (1) $\quad z^3 + 2\cdot\frac{2}{z} = 0 \Rightarrow z^4+4 = 0$

$|z| = 0$ $\quad z^4+4 = 0$

$\Rightarrow \mathbf{z = 0}$ $\quad \mathbf{z^4+4 = 0}$ **gives 4 solution**

∴ **It has five solution.**

Exercise 2.6 (2)

$z = x+iy$, show that locus of z , Im $\left(\frac{2z+1}{iz+1}\right) = 0$ is

$\mathbf{2x^2+2y^2+x-2y = 0}$

Solution : $\mathbf{z = x+iy}$

$$\frac{2z+1}{iz+1} = \frac{2(x+iy)+1}{i(x+iy)+1} = \frac{2x+i2y+1}{ix+i^2y+1}$$

$$= \frac{(2x+1)+i2y}{(1-y)+ix}\times\frac{(1-y)-ix}{(1-y)-ix}$$

$$= \frac{(2x+1)(1-y)-ix(2x+1)+i2y(1-y)}{(1-y)^2+x^2}$$

$$= \left[\frac{(2x+1)(1-y)+2xy}{(1-y)^2+x^2}\right] + i\left[\frac{2y(1-y)-x(2x+1)}{(1-y)^2+x^2}\right]$$

R.P $\qquad$ I.P

$$\text{Im}\left(\frac{2z+1}{iz+1}\right) = 0 \Rightarrow \frac{2y(1-y)-x(2x+1)}{(1-y)^2+x^2} = 0$$

$2y - 2y^2 - 2x^2 - x = 0$

∴ **Locus is** $\mathbf{2x^2+2y^2+x-2y = 0}$

Example 2.27:

$\mathbf{z = x+iy}$ $\quad$ **arg** $\left(\frac{z-1}{z+1}\right) = \frac{\pi}{2}$ ∴ **Locus is $x^2+y^2 = 1$.**

Solution : $\mathbf{z = x+iy}$

$$\frac{z-1}{z+1} = \frac{x+iy-1}{x+iy+1} = \frac{(x-1)+iy}{(x+1)+iy}\times\frac{(x+1)-iy}{(x+1)-iy}$$

$$= \frac{(x-1)(x+1)-iy(x-1)+iy(x+1)-i^2y^2}{(x+1)^2+y^2}$$

$$= \frac{(x-1)(x+1)+y^2}{(x+1)^2+y^2} + i\,\frac{y(x+1)-y(x-1)}{(x+1)^2+y^2}$$

R.P $\qquad$ I.P

$$\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{2} \Rightarrow \tan^{-1}\left[\frac{\frac{y(x+1)-y(x-1)}{(x+1)^2+y^2}}{\frac{(x-1)(x+1)+y^2}{(x+1)^2+y^2}}\right] = \frac{\pi}{2}$$

$$\Rightarrow \frac{y(x+1)-y(x-1)}{(x-1)(x+1)+y^2} = \tan\frac{\pi}{2} = \infty$$

$\Rightarrow$ **Dr** $= \mathbf{0}$ $\qquad$ **i.e** $\mathbf{(x-1)(x+1)+y^2 = 0}$

$\mathbf{x^2-1+y^2 = 0}$

$\Rightarrow \quad \mathbf{x^2+y^2 = 1}$

Exercise 2.7. (6)

If $z = x + iy$ arg $\left(\frac{z-i}{z+2}\right) = \frac{\pi}{4}$ S.T. $x^2 + y^2 + 3x - 3y + 2 = 0$.

Solution : $z = x + iy$

$$\frac{z-i}{z+2} = \frac{x+iy-i}{x+iy+2} = \frac{x+i(y-1)}{(x+2)+iy} = \frac{x+i(y-1)}{(x+2)+iy} \times \frac{(x+2)-iy}{(x+2)-iy}$$

$$= \frac{x(x+2)-ixy+i(y-1)(x+2)-i^2y(y-1)}{(x+2)^2+y^2}$$

$$= \frac{x(x+2)+y(y-1)}{(x+2)^2+y^2} + i\,\frac{(y-1)(x+2)-xy}{(x+2)^2+y^2}$$

$$\arg\left(\frac{z-i}{z+2}\right) = \frac{\pi}{4} \Rightarrow \tan^{-1}\left[\frac{\frac{(y-1)(x+2)-xy}{(x+2)^2+y^2}}{\frac{x(x+2)+y(y-1)}{(x+2)^2+y^2}}\right] = \frac{\pi}{4}$$

$$\frac{(y-1)(x+2)-xy}{x(x+2)+y(y-1)} = \tan\frac{\pi}{4} = 1$$

$$xy + 2y - x - 2 - xy = x^2 + 2x + y^2 - y$$

$$x^2 + y^2 + 2x - y - 2y + x + 2 = 0$$

Locus is $x^2 + y^2 + 3x - 3y + 2 = 0$

Exercise 2.7 (4): If $\frac{1+z}{1-z} = \cos 2\theta + i\sin 2\theta$ then $z = i\tan\theta$

Solution : $\frac{1+z}{1-z} = \cos 2\theta + i\sin 2\theta$

$$\frac{1+z}{1-z} = e^{i2\theta} \Rightarrow 1 + z = e^{i2\theta}(1-z) = e^{i2\theta} - ze^{i2\theta}$$

$$z + ze^{i2\theta} = e^{i2\theta} - 1 \Rightarrow z(1 + e^{i2\theta}) = e^{i2\theta} - 1$$

$z = \frac{e^{i2\theta}-1}{1+e^{i2\theta}}$ **divide nr & dr be $e^{i\theta}$**

$$z = \frac{e^{i\theta}-\frac{1}{e^{i\theta}}}{e^{i\theta}+\frac{1}{e^{i\theta}}} = \frac{e^{i\theta}-e^{-i\theta}}{e^{i\theta}+e^{-i\theta}} = \frac{\cos\theta+i\sin\theta-(\cos\theta-i\sin\theta)}{\cos\theta+i\sin\theta+\cos\theta-i\sin\theta}$$

$$z = \frac{\cos\theta+i\sin\theta-\cos\theta+i\sin\theta}{2\cos\theta} = \frac{2i\sin\theta}{2\cos\theta} \Rightarrow z = i\tan\theta$$

(6) $\cos\alpha + \cos\beta + \cos\gamma = \sin\alpha + \sin\beta + \sin\gamma$

S.T. $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha + \beta + \gamma$

$\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha + \beta + \gamma)$

Solution :

$\cos\alpha + \cos\beta + \cos\gamma = 0$

$i\sin\alpha + i\sin\beta + i\sin\gamma = 0i$

$\cos\alpha + \cos\beta + \cos\gamma + i\sin\alpha + i\sin\beta + i\sin\gamma = 0 + i0$ -- (A)

let $a = \cos\alpha + i\sin\alpha = e^{i\alpha}$: $b = \cos\beta + i\sin\beta = e^{i\beta}$

$c = \cos\gamma + i\sin\gamma = e^{i\gamma}$

From (A) we get $a + b + c = 0$

$\Rightarrow a^3 + b^3 + c^3 = 3abc$

$(e^{i\alpha})^3 + (e^{i\beta})^3 + (e^{i\gamma})^3 = 3e^{i\alpha}\cdot e^{i\beta}\cdot e^{i\gamma}$

$\Rightarrow e^{i3\alpha} + e^{i3\beta} + e^{i\beta\gamma} = 3e^{i(\alpha+\beta+\gamma)}$

$\cos 3\alpha + i\sin 3\alpha + \cos 3\beta + i\sin 3\beta + \cos 3\gamma + i\sin 3\gamma$
$= 3((\cos(\alpha+\beta+\gamma) + i\sin(\alpha+\beta+\gamma))$

$(\cos 3\alpha + \cos 3\beta + \cos 3\gamma) + i(\sin 3\alpha + \sin 3\beta + \sin 3\gamma)$

$= 3[\cos(\alpha+\beta+\gamma)] + \sin(\alpha+\beta+\gamma)]$

Equating real part

$\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha + \beta + \gamma)$

$\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha + \beta + \gamma)$

Example 2.34 : **solve : $z^3 + 8i = 0$**

$z^3 + 8i = 0$

$z^3 = -8i$

$= 8(-i)$

$$z^3 = 8\left[\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right]$$

$$= 8\left[\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right]$$

$$= 8\left[\cos\left(2k\pi - \frac{\pi}{2}\right) + i\sin\left(2k\pi - \frac{\pi}{2}\right)\right)$$

$$z^3 = 8\left[\cos\left(\frac{4k\pi-\pi}{2}\right) + i\sin\left(\frac{4k\pi-\pi}{2}\right)\right]$$

$$z = 8^{\frac{1}{3}}\left[\cos\left(\frac{4k\pi-\pi}{2}\right) + i\sin\left(\frac{4k\pi-\pi}{2}\right)\right]^{\frac{1}{3}}$$

$$z = (2^3)^{1/3}\left[\cos(4k-1)\frac{\pi}{6} + i\sin(4k-1)\frac{\pi}{6}\right]$$

$k = 0,1\,,2$

$$\mathbf{k = 0\,,\ z = 2\left[\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right] = 2\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)}$$

$$\mathbf{= \sqrt{3} - i}$$

$$\mathbf{k = 1, z = 2\left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right] = 2(0 + i) = 2i}$$

$$\mathbf{k = 2,\ z = 2\left[\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right] = \left(-\frac{\sqrt{3}}{2} - \frac{i}{2}\right)}$$

$$\mathbf{= -\sqrt{3} - i}$$

Example 2.35

Find the cube roots of $\sqrt{3} + i$

Solution :

Let $z = \sqrt[3]{\sqrt{3} + i} = (\sqrt{3} + i)^{\frac{1}{3}}$

$\sqrt{3} + i = r(\cos\theta + i\sin\theta)$

$a = \sqrt{3}$ $b = 1$ $a^2 = 3$ $b^2 = 1$

$r = \sqrt{a^2 + b^2} = \sqrt{3+1} = \sqrt{4} = 2$

$$\alpha = \tan^{-1}\left|\frac{b}{a}\right| = \tan^{-1}\left|\frac{1}{\sqrt{3}}\right| = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$\alpha = \frac{\pi}{6}$$

$\theta = \alpha = \frac{\pi}{6}$ $\sqrt{3} + i$ lies in I Quadrant

$$\sqrt{3} + i = 2\left[\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right]$$

$$= 2\left[\cos\left(2k\pi + \frac{\pi}{6}\right) + i\sin\left(2k\pi + \frac{\pi}{6}\right)\right]$$

$$= 2\left[\cos\left(\frac{12k\pi+\pi}{6}\right) + i\sin\left(\frac{12k\pi+\pi}{6}\right)\right]$$

$$(\sqrt{3} + i)^{1/3} = 2^{1/3}\left[\cos\left(\frac{12k\pi+\pi}{6}\right) + i\sin\left(\frac{12k\pi+\pi}{6}\right)\right]^{\frac{1}{3}}$$

$$z = 2^{1/3}\left[\cos(12k+1)\frac{\pi}{18} + i\sin(12k+1)\frac{\pi}{18}\right.$$

$k = 0,1,2$

$$\mathbf{k = 0\ \ z_1 = 2^{1/3}\left(\cos\frac{\pi}{18} + i\sin\frac{\pi}{18}\right) = 2^{1/3}e^{i\frac{\pi}{18}}}$$

$$\mathbf{k = 1\ \ z_2 = 2^{1/3}\left[\cos\frac{13\pi}{18} + i\sin\frac{13\pi}{18}\right] = 2^{1/3}e^{i\frac{13\pi}{18}}}$$

$$\mathbf{k = 2\ \ z_3 = 2^{1/3}\left[\cos\frac{25\pi}{18} + i\sin\frac{25\pi}{18}\right] = 2^{1/3}e^{i\frac{25\pi}{18}}}$$

(3) If $(x_1+iy_1)(x_2+iy_2)\dots(x_n+iy_n)=a+ib$

show that $(x_1^2+y_1^2)(x_2^2+y_2^2)\dots(x_n^2+y_n^2)=a^2+b^2$

$\sum_{r=1}^{n}\tan^{-1}\left(\frac{y_r}{x_r}\right)=k\pi+\tan^{-1}\left(\frac{b}{a}\right) \qquad k\in z$

<u>Solution :</u>

(i) $(x_1+iy_1)(x_2+iy_2)\dots(x_n+iy_n)=a+ib$

$|(x_1+iy_1)(x_2+iy_2)(x_3+iy_3)|=|a+ib|$

$|(x_1+iy_1)||(x_2+iy_2)|\dots|(x_3+iy_3)|=|a+ib|$

$\sqrt{x_1^2+y_1^2}\sqrt{x_2^2+y_2^2}\cdots\sqrt{x_n^2+y_n^2}=\sqrt{a^2+b^2}$

On squaring : $(x_1^2+y_1^2)(x_2^2+y_2^2)\dots(x_n^2+y_n^2)=a^2+b^2$

(ii) $\arg[(x_1+iy_1)(x_2+iy_2)\dots(x_n+iy_n)]=\arg(a+ib)$

$\arg(x_1+iy_1)+\arg(x_2+iy_2)+\cdots+\arg(x_n+iy_n)=\arg(a+ib)$

$\Rightarrow \tan^{-1}\left(\frac{y_1}{x_1}\right)+\tan^{-1}\left(\frac{y_2}{x_2}\right)+\cdots+\tan^{-1}\left(\frac{y_n}{x_n}\right)=\tan^{-1}\left(\frac{b}{a}\right)$

$\therefore$ Given solution

$$\tan^{-1}\left(\frac{y_1}{x_1}\right)+\tan^{-1}\left(\frac{y_2}{x_2}\right)+\cdots+\tan^{-1}\left(\frac{y_n}{x_n}\right)=k\pi+\tan^{-1}\left(\frac{b}{a}\right)$$

<u>Example : 2.36.</u>

z_1, z_2, z_3 are vertices of an equilateral traingle inscribed in the circle $|z|=2$, $z_1=1+i\sqrt{3}$.

<u>Solution :</u>

$|z|=2$ represents circle with center $(0,0)$ and radius $=2$

z_1,z_2,z_3 lies on circle and forms a vertices of equilateral triangle.

z_2,z_3 obtained by rotating $z_1=1+i\sqrt{3}$ by $120°, 240°$ in anti clockwise direction respectively.

$z_1=1+i\sqrt{3}$

$z_2=(1+i\sqrt{3})\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$

$=(1+i\sqrt{3})\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right) \quad =-\frac{1}{2}+i\frac{\sqrt{3}}{2}-\frac{i\sqrt{3}}{2}+i^2\frac{3}{2}$

$=-\frac{1}{2}-\frac{3}{2}=-\frac{4}{2}=-2$

z_3 is obtained by multiplying z_2 with $e^{i\frac{2\pi}{3}}$

$z_3=-2\left[\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right]$

$=-2\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)=\frac{2}{2}-i\,2\cdot\frac{\sqrt{3}}{2}$

$=1-i\sqrt{3}$

$\therefore z_2=-2\,;\ z_3=1-i\sqrt{3}$

$x+\frac{1}{x}=2\cos\alpha \Rightarrow x^2+1=2\cos\alpha x$

$x^2-2\cos\alpha x+\cos^2\alpha+\sin^2\alpha=0$

$(x-\cos\alpha)^2=-\sin^2\alpha \Rightarrow (x-\cos\alpha)^2=i^2\sin^2\alpha$

$x-\cos\alpha=\pm\sqrt{i^2\sin^2\alpha} \Rightarrow x=\cos\alpha\pm i\sin\alpha$

let $x=\cos\alpha+i\sin\alpha$

Similarly $y=\cos\beta+i\sin\beta$

4. If $2\cos\alpha=x+\frac{1}{x}$ & $2\cos\beta=y+\frac{1}{y}$

Show :

(i) $\frac{x}{y}+\frac{y}{x}=2\cos(\alpha-\beta)$

$\frac{x}{y}=\frac{\cos\alpha+i\sin\alpha}{\cos\beta+i\sin\beta}=\cos(\alpha-\beta)+i\sin(\alpha-\beta)$

$\frac{y}{x}=\left(\frac{x}{y}\right)^{-1}=[\cos(\alpha-\beta)+i\sin(\alpha-\beta)]^{-1}$

$=\cos(\alpha-\beta)-i\sin(\alpha-\beta)$

$\frac{x}{y}+\frac{y}{x}=\cos(\alpha-\beta)+i\sin(\alpha-\beta)+\cos(\alpha-\beta)-i\sin(\alpha-\beta)$

$=2\cos(\alpha-\beta)$

(ii) $xy-\frac{1}{xy}=2i\sin(\alpha+\beta)$

$xy=(\cos\alpha+i\sin\alpha)(\cos\beta+i\sin\beta)$

$=\cos(\alpha+\beta)+i\sin(\alpha+\beta)$

$\frac{1}{xy}=(xy)^{-1}=[\cos(\alpha+\beta)+i\sin(x+\beta)]^{-1}$

$=\cos(\alpha+\beta)-i\sin(\alpha+\beta)$

$xy-\frac{1}{xy}=\cos(\alpha+\beta)+i\sin(\alpha+\beta)$

$-\cos(\alpha+\beta)+i\sin(\alpha+\beta)$

$xy-\frac{1}{xy}=2i\sin(\alpha+\beta)$

(iii) $\frac{x^m}{y^n}-\frac{y^n}{x^m}=2i\sin(m\alpha-n\beta)$

$\frac{x^m}{y^n}=\frac{(\cos\alpha+i\sin\alpha)^m}{(\cos\beta+i\sin\beta)^n}=\frac{\cos m\alpha+i\sin m\alpha}{\cos n\beta+i\sin n\beta}$

$=\cos(m\alpha-n\beta)+i\sin(m\alpha-n\beta)$

$\frac{x^m}{y^n}=\left(\frac{x^m}{y^n}\right)^{-1}=[\cos(m\alpha-n\beta)+i\sin(m\alpha-n\beta)]^{-1}$

$=\cos(m\alpha-n\beta)-i\sin(m\alpha-n\beta)$

$\frac{x^m}{y^n}-\frac{y^n}{x^m}=\cos(m\alpha-n\beta)+i\sin(m\alpha-n\beta)-$

$\cos(m\alpha-n\beta)+i\sin(m\alpha-n\beta)$

$=2i\sin(m\alpha-n\beta)$

$x^my^n+\frac{1}{x^my^n}=2\cos(m\alpha+n\beta)$

$x^my^n=(\cos\alpha+i\sin\alpha)^m(\cos\beta+i\sin\beta)^n$

$=(\cos m\alpha+i\sin m\alpha)(\cos n\beta+i\sin n\beta)$

$=\cos(m\alpha+n\beta)+i\sin(m\alpha+n\beta)$

$\frac{1}{x^my^n}=(x^my^n)^{-1}$

$=[\cos(m\alpha+n\beta)+i\sin(m\alpha+n\beta)]^{-1}$

$=\cos(m\alpha+n\beta)-i\sin(m\alpha+n\beta)$

$x^my^n+\frac{1}{x^my^n}=\cos(m\alpha+n\beta)+i\sin(m\alpha+n\beta)$

$+\cos(m\alpha+n\beta)-i\sin(m\alpha+n\beta)$

$=2\cos(m\alpha+n\beta)$

CHAPTER 3 - THEORY OF EQUATION
2 MARKS & 3 MARKS

2 - MARKS

EXERCISE 3.1

2) (i) construct a cubic polynomial with roots $1, 2, 3$

Solution : $\alpha = 1 \quad \beta = 2 \quad \gamma = 3$

$\Sigma_1 = \alpha + \beta + \gamma = 1 + 2 + 3 = 6$

$\Sigma_2 = \alpha\beta + \alpha\gamma + \beta\gamma = 1(2) + 1(3) + 2(3)$

$= 2 + 3 + 6 = 11$

$\Sigma_3 = \alpha\beta\gamma = 1(2)(3) = 6.$

$\therefore$ Eqn: $x^3 - \Sigma_1 x^2 + \Sigma_2 x - \Sigma_3 = 0$

$\therefore x^3 - 6x^2 + 11x - 6 = 0.$

2(ii) roots $1, 1, -2$

solution: $\alpha = 1 \quad \beta = 1 \quad \gamma = -2$

$\Sigma_1 = \alpha + \beta + \gamma = 1 + 1 + (-2) = 2 - 2 = 0$

$\Sigma_2 = \alpha\beta + \alpha\gamma + \beta\gamma = 1(1) + 1(-2) + 1(-2)$

$= 1 - 2 - 2 = -3$

$\Sigma_3 = \alpha\beta\gamma = 1(1)(-2) = -2.$

$\therefore$ Equation: $x^3 - \Sigma_1 x^2 + \Sigma_2 x - \Sigma_3 = 0$

$\therefore x^3 - 0x^2 + (-3)x - (-2) = 0$ I.e $x^3 - 3x + 2 = 0$

2(iii) roots $2, 1/2$ and 1 .

solution: $\alpha = 2 \; \beta = \frac{1}{2}, \quad \gamma = 1.$

$$\Sigma_1 = \alpha + \beta + \gamma = 2 + \frac{1}{2} + 1 = \frac{4 + 1 + 2}{2} = \frac{7}{2}$$

$$\Sigma_2 = \alpha\beta + \alpha\gamma + \beta\gamma = 2\left(\frac{1}{2}\right) + 2(1) + \frac{1}{2}(1)$$

$$= 1 + 2 + \frac{1}{2} = \frac{2 + 4 + 1}{2} = \frac{7}{2}$$

$$\Sigma_3 = \alpha\beta\gamma = 2\left(\frac{1}{2}\right)(1) = 1.$$

$\therefore$ Equation: $x^3 - \Sigma_1 x^2 + \Sigma_2 x - \Sigma_3 = 0$

$\therefore x^3 - \left(\frac{7}{2}\right)x^2 + \frac{7}{2}(x) - 1 = 0 \Rightarrow 2x^3 - 7x^2 + 7x - 2 = 0$

---

8. If α , β , γ and δ are the roots of polynomial equation $2x^4 + 5x^3 - 7x^2 + 8 = 0$. find a quadratic equation with integer co efficients whose roots are $\alpha + \beta + \gamma + \delta$ and $\alpha\beta\gamma\delta$.

Solution: $2x^4 + 5x^3 - 7x^2 + 0x + 8 = 0$

$a = 2 \quad b = 5 \quad c = -7 \quad d = 0 \quad e = 8$

$\Sigma_1 = \alpha + \beta + \gamma + \delta = -b/a = -5/2$

$\Sigma_4 = \alpha\beta\gamma\delta = \frac{e}{a} = \frac{8}{2} = 4$

roots are $\alpha + \beta + \gamma + \delta$ and $\alpha\beta\gamma\delta$

i.e $-5/2$ and 4 .

$$\text{S.O.R} = -\frac{5}{2} + 4 = \frac{-5 + 8}{2} = \frac{3}{2}$$

$\text{P.O.R} = \frac{-5}{2}(4) = -10$

Eqn : $x^2 - (\text{S.O.R})x + \text{P.O.R} = 0$

$\Rightarrow x^2 - \frac{3}{2}x + (-10) = 0 \Rightarrow 2x^2 - 3x - 20 = 0$.

---

(11) A 12 metre tall tree was broken into two parts . It was found that the height of the part which was left standing was the cube root of length of the part that was cut away. Formulate this into a mathematical Problem to find the height of part which was left standing.

Solution :

let $AC = 12$, $AB = x$, $BC = 12 - x$

Given: $x = \sqrt[3]{12 - x}$

$\Rightarrow x^3 = 12 - x \Rightarrow x^3 + x - 12 = 0$

EXERCISE 3.2

2) Find a polynomial equation of minimum deqree with rational co.efficients having $2 + \sqrt{3}i$ as a root

Solution : Given $2 + \sqrt{3}i$ is a root $\therefore$ other root is $2 - \sqrt{3}i$

$\text{S.O.R} = 2 + \sqrt{3}i + 2 - \sqrt{3}i = 4$

$\text{P.O.R} = (2 + \sqrt{3}i)(2 - \sqrt{3}i) = 2^2 + \sqrt{3}^2 = 4 + 3 = 7$

$\therefore$ Eqn is $\quad x^2 - (\text{S.O.R})x + \text{P.O.R} = 0$

$\Rightarrow \quad x^2 - 4x + 7 = 0.$

---

2) Find a polynomial equation with minimum degree with rational co.efficients having $2i + 3$ as a root

Solution : Given one root is $2i + 3 = 3 + 2i$

Other root is $3 - 2i$

Sum of the roots $= 3 + 2i + 3 - 2i = 6$

Product of roots $= (3 + 2i)(3 - 2i)$

$= 3^2 + 2^2 = 9 + 4 = 13$

Equation is $\quad x^2 - (\text{S.O.R})x + \text{P.O.R} = 0$

$\Rightarrow \quad x^2 - 6x + 13 = 0$

EXERCISE 3.3

(7) Solve the equation : $x^4 - 14x^2 + 45 = 0$

$(x^2)^2 - 14x^2 + 45 = 0 \quad$ let $t = x^2$

$t^2 - 14t + 45 = 0 \Rightarrow (t - 9)(t - 5) = 0$

$t - 9 = 0 \quad t - 5 = 0$

$t = 9 \quad t = 5$

$\Rightarrow x^2 = 9 \quad \Rightarrow x^2 = 5$

$\Rightarrow x = \pm\sqrt{9} \quad \Rightarrow x = \pm\sqrt{5}$

$\Rightarrow x = \pm 3 \quad$ Roots are $3 , -3 , \sqrt{5} , -\sqrt{5}$

EXERCISE 3.5: **2(ii) $x^8 - 3x + 1 = 0$**

Solution :

$a_n = 1 \quad a_0 = 1$. Note : $\frac{p}{q}$ is a rational root of p(x) (p,q)=1

Then p is divisor 1 , $\quad$ q is divisor 1

possible values of p ± 1, Possible values of q ± 1

$\frac{p}{q}$ possible value is $\pm\frac{1}{1}$

$p(x) = x^8 - 3(x) + 1$

$p(1) = 1 - 3(1) + 1 = -1 \neq 0$

$p(-1) = (-1)^8 - 3(-1) + 1 = 1 + 3 + 1 \neq 0$

$\therefore$ no rational roots

EXERCISE 3.6

1. Discuss the maximum possible number of positive and negative roots of

$9x^9 - 4x^8 + 4x^7 - 3x^6 + 2x^5 + x^3 + 7x^2 + 7x + 2 = 0$

Solution :

$p(x) = 9x^9 - 4x^8 + 4x^7 - 3x^6 + 2x^5 + x^3 + 7x^2 + 7x + 2$

$+ \quad - \quad + \quad - \quad + \quad + \quad + \quad + \quad +$

No of sign changes $= 4$ $\quad\therefore$ max no of positive real roots $= 4$

$p(-x) = 9(-x)^9 - 4(-x)^8 + 4(-x)^7 - 3(-x)^6 + 2(-x)^5 + (-x)^3 + 7(-x)^2 + 7(-x) + 2$

$= -9x^9 - 4x^8 - 4x^7 - 3x^6 - 2x^5 - x^3 + 7x^2 - 7x + 2$

$- \quad - \quad - \quad - \quad - \quad - \quad + \quad - \quad +$

No of sign changes $= 3$

Max no of negative real roots $= 3$

Degree $= 9$

min. no of complex roots $= 9 - (4 + 3) = 9 - 7 = 2$

2) show that the equation $x^9-5x^5+4x^4+2x^2+1=0$ has atleast 6 imaginary solution.

Solution :

Let $p(x)=x^9-5x^5+4x^4+2x^2+1$

$+ \quad - \quad + \quad + \quad +$

No of sign change in $p(x)=2$

Max no of positive real roots $=2$

$p(-x)=(-x)^9-5(-x)^5+4(-x)^4+2(-x)^2+1$

$=-x^9+5x^5+4x^4+2x^2+1$

$- \quad + \quad + \quad + \quad +$ No of sign changes $=1$

Max no of negative real roots $=1$.

degree $=9$

Max no of complex roots $=9-(2+1)=9-3=6$

---

2) **Determine the number of +ive and negative roots of the equation $x^9-5x^8-14x^7=0$**

Solution :

Let $p(x)=x^9-5x^8-14x^7$

$+ \quad - \quad -$

Number of sign change in $p(x)$ is 1

$\therefore$ p(x) has max 1 the real roots .

$p(-x)=(-x)^9-5(-x)^8-14(-x)^7$

$=-x^9-5x^8+14x^7$

$- \quad - \quad +$ Number of sign change in $p(-x)$ is 1

$\therefore$ p(x) has max 1 negative real roots .

Degree $=9$

min number of imaginary roots $=9-(1+1)=9-2=7$

---

3) Find the no of real zeros and imaginary of the polynomial $x^9+9x^7+7x^5+5x^3+3x$

Solution :

$p(x)=x^9+9x^7+7x^5+5x^3+3x$ (no sign change)

$p(-x)=-x^9-9x^7-7x^5-5x^3-3x$ (no sign change)

$\therefore$ There is no +ive & no -ive real roots

But $x=0$ is a root

no of unreal or imaginary roots = 9-1=8

---

3 - MARKS

EXERCISE 3.1

3). If α,β,γ are the roots of cubic equation $x^3+2x^2+3x+4=0$. Form the cubic equation whose roots are

(i) $2\alpha,2\beta,2\gamma$ (ii) $\frac{1}{\alpha},\frac{1}{\beta},\frac{1}{\gamma}$ (iii) $-\alpha,-\beta,-\gamma$

Solution: $x^3+2x^2+3x+4=0$

$a=1 \quad b=2 \quad c=3 \quad d=4$

$\alpha+\beta+\gamma=-\frac{b}{a}=-\frac{2}{1}=-2$

$\alpha\beta+\alpha\gamma+\beta\gamma=c/a=\frac{3}{1}=3$

$\alpha\beta\gamma=-d/a=-4/1=-4$

(i) roots are $2\alpha,2\beta,2\gamma$.

$\Sigma_1=2\alpha+2\beta+2\gamma=2(\alpha+\beta+\gamma)=2(-2)=-4$

$\Sigma_2=(2\alpha)(2\beta)+(2\alpha)(2\gamma)+(2\beta)(2\gamma)$

$=4\alpha\beta+4\alpha\gamma+4\beta\gamma$

$=4(\alpha\beta+\alpha\gamma+\beta\gamma)=4(3)=12$

$\Sigma_3=(2\alpha)(2\beta)(2\gamma)=8\alpha\beta\gamma=8(-4)=-32$

Equation : $x^3-\Sigma_1x^2+\Sigma_2x-\Sigma_3=0$

$x^3-(-4)x^2+12x-(-32)=0$

$\Rightarrow x^3+4x^2+12x+32=0$

(ii) roots are $\frac{1}{\alpha},\frac{1}{\beta},\frac{1}{\gamma}$

$\Sigma_1=\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{\beta\gamma+\alpha\gamma+\alpha\beta}{\alpha\beta\gamma}=\frac{3}{-4}=\frac{-3}{4}$

$\Sigma_2=\frac{1}{\alpha}.\frac{1}{\beta}+\frac{1}{\beta}.\frac{1}{\gamma}+\frac{1}{\alpha\gamma}=\frac{\gamma+\alpha+\beta}{\alpha\beta\gamma}=\frac{2}{4}$

$\Sigma_3=\frac{1}{\alpha}\cdot\frac{1}{\beta}\cdot\frac{1}{\gamma}=\frac{1}{\alpha\beta\gamma}=\frac{1}{-4}=\frac{-1}{4}$

Equation : $x^3-\Sigma_1x^2+\Sigma_2x-\Sigma_3=0$

$x^3-\left(\frac{-3}{4}\right)x^2+\frac{2}{4}x-\left(-\frac{1}{4}\right)=0 \quad \Rightarrow 4x^3+3x^2+2x+1=0.$

(iii) roots are $-\alpha,-\beta,-r$

$\Sigma_1=(-\alpha)+(-\beta)+(-\gamma)=-(\alpha+\beta+\gamma)=-(-2)=2$

$\Sigma_2=(-\alpha)(-\beta)+(-\alpha)(-\gamma)+(-\beta)(-\gamma)$

$=\alpha\beta+\alpha\gamma+\beta\gamma=3$

$\Sigma_3=(-\alpha)(-\beta)(-\gamma)=-\alpha\beta\gamma=-(-4)=4$

Equation : $x^3-\Sigma_1x^2+\Sigma_2x-\Sigma_3=0$

$x^3-(+2)x^2+3x-4=0$

$x^3-2x^2+3x-4=0.$

---

5)Find the sum of the squares of the roots of the equation $2x^4-8x^3+6x^2-3=0$

Solution: $2x^4-8x^3+6x^2+0x-3=0$

let the roots be $\alpha,\beta,\gamma,\delta$

$a=2 \quad b=-8 \quad c=6 \quad d=0 \quad e=-3$

$\Sigma_1=\alpha+\beta+\gamma+\delta=\frac{-b}{a}=\frac{-(-8)}{2}=4$

$\Sigma_2=\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta=\frac{c}{a}=\frac{6}{2}=3.$

$\alpha^2+\beta^2+\gamma^2+\delta^2=(\alpha+\beta+\gamma+\delta)^2-2(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)$

$\Rightarrow \alpha^2+\beta^2+\gamma^2+\delta^2=(4)^2-2(3)=16-6=10.$

---

(7) If α,β,γ are the roots of the equation $ax^3+bx^2+cx+d=0$, find $\sum\frac{\alpha}{\beta\gamma}$

Solution:

$ax^3+bx^2+cx+d=0$

let the Roots be α,β,γ

$\Sigma_1=\alpha+\beta+\gamma=-b/a$

$\Sigma_2=\alpha\beta+\beta\gamma+\alpha\gamma=c/a$

$\Sigma_3=\alpha\beta\gamma=-d/a$

To find: $\sum\frac{\alpha}{\beta\gamma}=\frac{\alpha}{\beta\gamma}+\frac{\beta}{\alpha\gamma}+\frac{\gamma}{\alpha\beta}=\frac{\alpha^2+\beta^2+\gamma^2}{\alpha\beta\gamma}$

$=\frac{(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\alpha\gamma)}{\alpha\beta\gamma}=\frac{\left(-\frac{b}{a}\right)^2-2.\frac{c}{a}}{-\frac{d}{a}}=\frac{\frac{b^2}{a^2}-\frac{2c}{a}}{-\frac{d}{a}}$

$=\frac{b^2-2ac}{-a^2}\times\frac{a}{d}=\frac{2ac-b^2}{ad}$

---

8. If p and q are the roots of the equation $lx^2+nx+n=0$ show that $\sqrt{\frac{p}{q}}+\sqrt{\frac{q}{p}}+\sqrt{\frac{n}{l}}=0$.

Solution : $lx^2+nx+n=0$

$a=l, b=n, c=n$ roots are p , q

$\Rightarrow p+q=\frac{-b}{a}=\frac{-n}{l}$

Also $pq=\frac{c}{a}=\frac{n}{l}$

Now , $\sqrt{\frac{p}{q}}+\sqrt{\frac{q}{p}}+\sqrt{\frac{n}{l}}=\frac{\sqrt{p}}{\sqrt{q}}+\frac{\sqrt{q}}{\sqrt{p}}+\sqrt{\frac{n}{l}}=\frac{p+q}{\sqrt{pq}}+\sqrt{\frac{n}{l}}$

$=\frac{-n/l}{\sqrt{\frac{n}{l}}}+\sqrt{\frac{n}{l}}=\frac{-\sqrt{\frac{n}{l}}\sqrt{\frac{n}{l}}}{\sqrt{\frac{n}{l}}}+\sqrt{\frac{n}{l}}=-\sqrt{\frac{n}{l}}+\sqrt{\frac{n}{l}}=0$

10. If the equation $x^2+px+q=0$ & $x^2+p'x+q'=0$ have a common root Show that it is $\frac{pq'-q'p}{q-q'}$ or $\frac{q-q'}{p'-p}$.

<u>Solution :</u>

$x^2+px+q=0 \qquad x^2+p'x+q'=0$

Let α be the common root

$\alpha^2+p\alpha+q=0 \quad \& \quad \alpha^2+p'\alpha+q'=0$

$\Rightarrow \dfrac{\alpha^2}{pq'-p'q}=\dfrac{\alpha}{q-q'}=\dfrac{1}{p'-p}$

$\Rightarrow \dfrac{\alpha^2}{\alpha}=\dfrac{pq'-p'q}{q-q'} \quad \& \quad \alpha=\dfrac{q-q'}{p'-p}$

$\Rightarrow \alpha=\frac{pq'-p'q}{q-q'}$ or $\frac{q-q'}{p'-p}$.

EXERCISE 3.2:

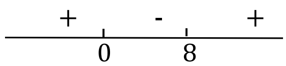
1} If k is real, discuss the nature of the roots of the polynomial equation $2x^2+kx+k=0$ interm of k.

<u>Solution :</u>

$2x^2+kx+k=0 \quad a=2 \;\; b=k \;\; c=k$

$\Delta=b^2-4ac=k^2-4(2)k=k^2-8k$

$=k(k-8)$



(i) for real and equal roots

$\Delta=0 \;\; \Rightarrow k(k-8)=0 \;\; k=0 \;\; k=8$

(ii) For real & distinct roots

$\Delta>0 \quad \Rightarrow k(k-8)>0$

$\Rightarrow k\in(-\infty,0)\cup(8,\infty)$

(iii) For imaginary roots

$\Delta<0 \quad \Rightarrow k(k-8)<0$

$\Rightarrow k\in(0,8)$

5) Prove that a straight line and parabola cannot intersect at more than 2 points

<u>Solution:</u>

Parabola eqn : $y^2=4ax$ – (1)

line eqn : $\quad y=mx+c$ – (2)

sub (2) in (1) $(mx+c)^2=4ax$

$\Rightarrow m^2x^2+2mcx+c^2=4ax$

$\Rightarrow m^2x^2+(2mc-4a)x+c^2=0$

This is a quadratic eqn in x , x can have max 2 values .

EXERCISE : 3.3

1. Solve the cubic equation $x^3-x^2-18x+9=0$, if sum of the two of roots vanishes.

<u>Solution :</u>

$2x^3-x^2-18x+9=0$

$a=2 \quad b=-1 \quad c=-18 \quad d=9$

Let the roots be α , β , γ

given $\alpha+\beta=0$

also $\alpha+\beta+\gamma=\frac{-b}{a}=\frac{-(-1)}{2}=\frac{1}{2}$

$\because \alpha+\beta=0 \quad \Rightarrow \gamma=1/2$

$\frac{1}{2}$	2	-1	-18	9
	0	1	0	-9
	2	0	-18	0

Quad. eqn $2x^2-18=0$

$\Rightarrow \quad x^2-9=0$

$\Rightarrow \quad x^2=9$

$\Rightarrow \quad x=\pm 3$

roots are $3,-3,\frac{1}{2}$.

2. Solve the equation $9x^3-36x^2+44x-16=0$ if the roots form an arithmatic progression

<u>Solution :</u> $\quad 9x^3-36x^2+44x-16=0$

$a=9 \quad b=-36 \;\; c=44 \;\; d=-16$

Let the roots be α , β , γ be in . A.P

$\alpha=a_1-d$, $\beta=a_1$, $r=a_1+d$

$S.O.R=a_1-d+a_1+a_1+d=\dfrac{-b}{a}=\dfrac{-(-36)}{9}$

$\Rightarrow 3a_1=4 \quad \Rightarrow a_1=4/3$

$\frac{4}{3}$	9	-36	44	-16
	0	12	-32	16
	9	-24	12	0

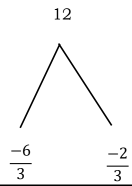
Quadratic equation is

$9x^2-24x+12=0$

$\div 3 \quad 3x^2-8x+4=0$

$(x-2)\left(x-\frac{2}{3}\right)=0 \quad \therefore x=2 , 2/3$

$\therefore$ Roots are $2,\ \frac{2}{3},\ \frac{4}{3}$



2. Solve the equation $3x^3-26x^2+52x-24=0$ if the roots form a geometric progression .

<u>Solution :</u> $\quad 3x^3-26x^2+52x-24=0$

$a=3 \quad b=-26 \quad c=52 \quad d=-24$

let the roots be α,β,γ .root are in G.P. $\alpha=\frac{a_1}{r} \quad \beta=a_1 \;\; \gamma=a_1r$

Product of roots $=\frac{a_1}{r}\cdot a_1\cdot a_1r=\frac{-(d)}{a}$

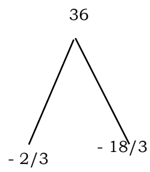
$\Rightarrow a_1^3=\dfrac{-(-24)}{3}=\dfrac{24}{3} \quad \Rightarrow \quad a_1^3=8 \Rightarrow \quad a_1=2$

2	3	-26	52	-24
	0	6	-40	24
	3	-20	12	0

Quadratic eqn $3x^2-20x+12=0$

$\Rightarrow \quad \left(x-\frac{2}{3}\right)(x-6)=0$

$\Rightarrow x=\frac{2}{3},6 \;\; \therefore$ roots are $\frac{2}{3},2,6$



6.Solve the equation

<u>Solution :</u>

(i) $2x^3-9x^2+10x=3$, $\quad \mathbf{2+(-9)+10+(-3)=12-12=0}$

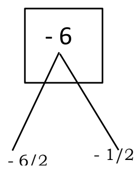
$2x^3-9x^2+10x-3=0$ (∴ x =1 is a root)

1	2	-9	10	-3
	0	2	-7	3
	2	-7	3	0

Quadratic eqn $2x^2-7x+3=0$

$(x-3)\left(x-\frac{1}{2}\right)=0 \;\Rightarrow\; x=3 , x=\frac{1}{2}$

Roots are $1,3,\frac{1}{2}$



(ii) $8x^3-2x^2-7x+3=0 \qquad \mathbf{8+(-2)+(-7)+3=11-9=2\neq 0}$

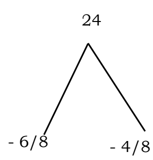
But $8+(-7)=1 \qquad -2+3=1$ (x = -1 is a root)

-1	8	-2	-7	3
	0	-8	10	-3
	8	-10	3	0

Quadratic equation .

$8x^2-10x+3=0 \Rightarrow \left(x-\dfrac{3}{4}\right)\left(x-\dfrac{1}{2}\right)=0$

$x=\frac{3}{4} \quad x=\frac{1}{2} \qquad$ Roots are $-1,\frac{3}{4},\frac{1}{2}$



Exercise 3.5

1 (i) Solve : $\sin^2 x - 5\sin x + 4 = 0$.

Solution :

$\sin^2 x - 5\sin x + 4 = 0$

Let $t = \sin x$

$\therefore\ t^2 - 5t + 4 = 0$

$(t-4)(t-1) = 0$

$\Rightarrow t - 4 = 0 \qquad t - 1 = 0$

$\Rightarrow t = 4 \qquad t = 1$

$\Rightarrow \sin x = 4 \qquad \sin x = 1$

Not possible $\quad \sin x = \sin\frac{\pi}{2}$

$\Rightarrow x = n\pi + (-1)^n \frac{\pi}{2} \quad n \in Z$

---

(1) (ii): Solve: $12x^3 + 8x = 29x^2 - 4$

Solution :

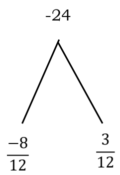
$12x^3 + 8x = 29x^2 - 4 \Rightarrow 12x^3 - 29x^2 + 8x + 4 = 0$

1 and -1 are not roots of above equation

2	12	-29	8	4
	0	24	-10	-4
	12	-5	-2	0

Quad. eqn $12x^2 - 5x - 2 = 0$

$\left(x - \frac{2}{3}\right)\left(x + \frac{1}{4}\right) = 0$



$x - \frac{2}{3} = 0 \qquad x + \frac{1}{4} = 0$

$x = \frac{2}{3}, -\frac{1}{4}$

roots are 2 , $\frac{2}{3}, -\frac{1}{4}$.

---

2 (i).Examine for the rational roots of $2x^3 - x^2 - 1 = 0$

Solution :

$a_n = 2 \quad a_0 = -1 \quad$ Rational root theorem ,

$\frac{p}{q}$ is a root of polynomial $(p, q) = 1$

p must divide a_0 , q must divide a_n

$a_0 = -1$ divisor of a_0 is $-1, 1$.

$a_n = 2$ divisor of a_n is $1, -1, 2, -2$

$\frac{p}{q}$ possible values $\pm\frac{1}{1}, \pm\frac{1}{2}$

1	2	-1	0	-1
	0	2	1	1
	2	1	1	0

$2x^2 + x + 1 = 0 \qquad X = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\Rightarrow x = \frac{-1 \pm \sqrt{1 - 4(1)(2)}}{2} = \frac{-1 \pm \sqrt{1-8}}{2}$

$= \frac{-1 \pm \sqrt{-7}}{2} = \frac{-1 \pm \sqrt{7}i}{2}$

$x = 1$ is the only rational root

---

(3): Solve: $8x^{3/2n} - 8x^{-3/2n} = 63$

Solution :

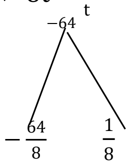
$8x^{3/2n} - 8 \cdot x^{-3/2n} = 63$

Let $t = x^{3/2n} \Rightarrow 8t - 8 \cdot t^{-1} = 63 \Rightarrow 8t - \frac{8}{t} = 63$

$\Rightarrow 8t^2 - 8 = 63t$

$\Rightarrow 8t^2 - 63t - 8 = 0$

$\Rightarrow (t-8)\left(t + \frac{1}{8}\right) = 0$



$\Rightarrow t - 8 = 0 \qquad t + \frac{1}{8} = 0$

$\Rightarrow t = 8 \qquad t = -\frac{1}{8}$

$\Rightarrow x^{3/2n} = 2^3 \qquad x^{3/2n} = \left(\frac{-1}{2}\right)^3$

$\Rightarrow x = (2^3)^{\frac{2n}{3}} \qquad x = \left[\left(\frac{-1}{2}\right)^3\right]^{\frac{2n}{3}}$

$= 2^{2n} = 4^n \qquad x = \left(\frac{-1}{2}\right)^{2n} = \frac{1}{4^n}$

---

(4): Solve : $2\sqrt{\frac{x}{a}} + 3\sqrt{\frac{a}{x}} = \frac{b}{a} + \frac{6a}{b}$.

Solution : $\quad 2\sqrt{\frac{x}{a}} + 3\sqrt{\frac{a}{x}} = \frac{b}{a} + \frac{6a}{b}$

$2t + 3 \cdot \frac{1}{t} = \frac{b}{a} + \frac{6a}{b} \qquad t = \sqrt{\frac{x}{a}}$

$\Rightarrow 2t^2 + 3 = \left(\frac{b}{a} + \frac{6a}{b}\right) \cdot t \qquad \frac{1}{t} = \sqrt{\frac{a}{x}}$

$\Rightarrow 2t^2 - \left(\frac{b}{a} + \frac{6a}{b}\right)t + 3 = 0$

$\Rightarrow 2t^2 - \frac{b}{a}t - \frac{6a}{b}t + 3 = 0 \Rightarrow t\left(2t - \frac{b}{a}\right) - \frac{3a}{b}\left(2t - \frac{b}{a}\right) = 0$

$\Rightarrow \left(t - \frac{3a}{b}\right)\left(2t - \frac{b}{a}\right) = 0 \Rightarrow t - \frac{3a}{b} = 0 \qquad 2t - \frac{b}{a} = 0$

$\Rightarrow t = \frac{3a}{b} \qquad 2t = \frac{b}{a}$

$\Rightarrow \sqrt{\frac{x}{a}} = \frac{3a}{b} \qquad \sqrt{\frac{x}{a}} = \frac{b}{2a}$

$\Rightarrow \frac{x}{a} = \frac{9a^2}{b^2} \qquad \Rightarrow \frac{x}{a} = \frac{b^2}{4a^2}$

$\Rightarrow x = \frac{9a^3}{b^2} \qquad \Rightarrow x = \frac{b^2}{4a} \qquad \therefore$ soln : $\frac{9a^3}{b^2}, \frac{b^2}{4a}$

5(ii) Solve: $x^4 + 3x^3 - 3x - 1 = 0$

Solution : $\quad x^4 + 3x^3 - 3x - 1 = 0$

1	1	3	0	-3	-1
	0	1	4	4	1
-1	1	4	4	1	0
	0	-1	-3	-1	
	1	3	1	0	

Quadralic eqn: $\quad x^2 + 3x + 1 = 0 \quad \left[\ X = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\ \right]$

$x = \frac{-3 \pm \sqrt{3^2 - 4(1)(1)}}{2} = \frac{-3 \pm \sqrt{9-4}}{2} = \frac{-3 \pm \sqrt{5}}{2}$

roots are $1, -1, \frac{-3+\sqrt{5}}{2}, \frac{-3-\sqrt{5}}{2}$

---

(6): Find all the real numbers satisfying

$4^x - 3(2^{x+2}) + 2^5 = 0$

Solution : $\quad 4^x - 3(2^{x+2}) + 2^5 = 0$

$(2^2)^x - 3 \cdot 2^x \cdot 2^2 + 32 = 0$

$(2^x)^2 - 3(4) \cdot 2^x + 32 = 0$

Let $2^x = t \Rightarrow t^2 - 12t + 32 = 0$

$\Rightarrow (t-8)(t-4) = 0$

$\Rightarrow t - 8 = 0 \qquad t - 4 = 0$

$\Rightarrow t = 8 \qquad t = 4$

$\Rightarrow 2^x = 2^3 \qquad 2^x = 2^2$

$\Rightarrow x = 3 \qquad x = 2$

CHAPTER 5 - 2 DIMENSIONAL ANALYTICAL GEOMETRY (ONLY 5 MARKS)

Exercsie 5.1 (6).

Find the equation of the circle through the points $(1,0), (-1,0)$, and $(0,1)$.

Solution:

Let the required circle be

$x^2 + y^2 + 2gx + 2fy + c = 0$ ---(A)

The circle passes through $(1,0), (-1,0)$ and $(0,1)$

$(1,0) \Rightarrow 1 + 0 + 2g(1) + 2f(0) + c = 0$

$2g + c = -1$............(1)

$(-1,0) \Rightarrow 1 + 0 + 2g(-1) + 2f(0) + c = 0$

$-2g + c = -1$........(2)

$(0,1) \Rightarrow 0 + 1 + 2g(0) + 2f(1) + c = 0$

$2f + c = -1$..........(3)

Now solving (1), (2) and (3) .

$2g + c = -1$ ----(1)

$-2g + c = -1$ -----(2)

(1) +(2) $\Rightarrow 2c = -2 \Rightarrow c = -1$

Substituting $c = -1$ in (1) we get

$2g - 1 = -1$

$2g = -1 + 1 = 0 \Rightarrow g = 0$

Substituting $c = -1$ in (3) we get

$2f - 1 = -1 \Rightarrow 2f = -1 + 1 = 0 \Rightarrow f = 0$

So we get $g = 0, f = 0$ and $c = -1$

So the required circle will be

$x^2 + y^2 + 2(0)x + 2(0)y - 1 = 0$

(i.e) $x^2 + y^2 - 1 = 0 \Rightarrow x^2 + y^2 = 1$

Example 5.10

Find the equation of the circle passing through the points $(1,1), (2,-1)$, and $(3,2)$.

Solution

Let the general equation of the circle be

$x^2 + y^2 + 2gx + 2fy + c = 0.$ ----(1)

It passes through points $(1,1), (2,-1)$ and $(3,2)$.

Therefore, $2g + 2f + c = -2$ ----(2)

$4g - 2f + c = -5$ -----(3)

$6g + 4f + c = -13$ ----(4)

(2)-(3) gives $-2g + 4f = 3$ ---(5)

(4)-(3) gives $2g + 6f = -8$ ---(6)

(5) + (6) gives $f = -\frac{1}{2}$

Substituting $f = -\frac{1}{2}$ in (6),$g = -\frac{5}{2}$

Substituting $f = -\frac{1}{2}$ and $g = -\frac{5}{2}$ in (2),c=4

Therefore, the required equation of the circle is

$x^2 + y^2 + 2\left(-\frac{5}{2}\right)x + 2\left(-\frac{1}{2}\right)y + 4 = 0$

$\Rightarrow x^2 + y^2 - 5x - y + 4 = 0$

Example 5.17:

Find the vertex , focus , directrix, and length of Latus rectum of $x^2 - 4x - 5y - 1 = 0$

Solution :

$x^2 - 4x - 5y - 1 = 0$

$x^2 - 4x = 5y + 1$

$x^2 - 4x + 4 = 5y + 4 + 1$

$(x-2)^2 = 5y + 5$

$(x-2)^2 = 5(y+1)$

$X = x - 2 \quad Y = y + 1 \quad 4a = 5 \Rightarrow a = \frac{5}{4}$

$X^2 = 5y$ Parabola open upward .

vertex = $(2,-1)$=(h,k) { x-2 = 0; y+1 = 0}

Focus : (0,a)⇒[(h,k+a)] $= \left(2, -1 + \frac{5}{4}\right) = \left(2, \frac{1}{4}\right)$

Eqn of directrix: Y= -a [y = k -a]

$y = -1 - \frac{5}{4} = -\frac{9}{4}$

$y = -\frac{9}{4}$

Length of Latus rectum = 4a = 5

Exercsie 5.2 - 4(iv)

Find the vertex , focus , directrix, and length of Latus rectum of $x^2 - 2x + 8y + 17 = 0$

Solution :

$x^2 - 2x + 8y + 17 = 0$

$x^2 - 2x = -8y - 17$

$x^2 - 2x + 1 = -8y - 17 + 1$

$(x-1)^2 = -8y - 16$

$(x-1)^2 = -8(y+2)$

$X = x - 1 \quad Y = y + 2 \quad 4a = 8 \Rightarrow a = 2$

$X^2 = -8Y$ Parabola open downward

Vertex(0,0) = (1,-2)=(h,k) $\{x - 1 = 0, \ y + 2 = 0\}$

Focus (0, -a)
$(h + 0, k - a)$ $= (1, -4)$

Equation of Latusrectum (Y = -a) :

$y + 2 = -2 \Rightarrow \quad y = -4$

Equation of directrix Y = a : $y + 2 = 2 \Rightarrow y = 0$

Length of latus rectum 4a = 8

Ex 5.2 - 4(v)

Find the vertex, focus, directrix and length of Latus rectum of $y^2 - 4y - 8x + 12 = 0$

Solution :

$y^2 - 4y - 8x + 12 = 0$

$y^2 - 4y = 8x - 12$

$y^2 - 4y + 4 = 8x - 12 + 4$

$(y-2)^2 = 8x - 8$

$(y-2)^2 = 8(x-1)$

$Y^2 = 8X$

$X = x - 1 \quad Y = y - 2 \quad 4a = 8 \Rightarrow a = 2$

Parabola open left ward

Vertex(0,0) = (1,2)=(h,k) $\{x - 1 = 0, x = -1;$
$y - 2 = 0, y = 2\}$

Focus (a, 0)
$(h + a, k + 0)$ = (3,2) {h + a = 1+2, k + 0 = 2+0 }

Eqn of directrix : X =- a

$x - 1 = -2 \quad x = -2 + 1 = -1 \quad x = -1$

Length of latus rectum 4a = 8

EXAMPLE 5.20

Find the vertex, focus, length of major and minor axis of $4x^2+36y^2+40x-288y+532=0$

Solution :

$4x^2+36y^2+40x-288y+532=0$

$4x^2+40x+36y^2-288y=-532$

$4(x^2+10x)+36(y^2-8y)=-532$

$4(x^2+10x+25)+36(y^2-8y+16)=-532+100+576$

$4(x+5)^2+36(y-4)^2=144$

$\div 144 \quad \frac{4(x+5)^2}{144}+\frac{36(y-4)^2}{144}=1$

$\frac{(x+5)^2}{36}+\frac{(y-4)^2}{4}=1$ Major axis X-axis:

$X=x+5 \qquad Y=y-4$

$\frac{x^2}{36}+\frac{y^2}{4}=1$, $\{a^2=36 \Rightarrow a=6 \quad b^2=4 \Rightarrow b=2\}$

$c^2=a^2-b^2=36-4=32$, $c=\sqrt{32}$

center $(-5,4)$

Foci: $(h\pm c,k)=(-5\pm 4\sqrt{2},4)$

i.e. $(-5+4\sqrt{2},4)$; $(-5-4\sqrt{2},4)=$

vertices$(h\pm a,k)$: $(-5\pm 6,4)$ i.e $(1,4)$; $(-11,4)$

length of major axis $=2a=2(6)=12$

length of minor axis $=2b=2(2)=4$.

EXAMPLE 5.21

For the ellipse $4x^2+y^2+24x-2y+21=0$

Find center , vertices, foci. Also prove $L\cdot L\cdot R=2$

Solution :

$4x^2+y^2+24x-2y+21=0$; $4x^2+24x+y^2-2y=-21$

$4(x^2+6x)+1(y^2-2y)=-21$

$4(x^2+6x+9)+1(y^2-2y+1)=-21+36+1=16$

$4(x+3)^2+(y-1)^2=16 \Rightarrow \div 16 \quad \frac{4(x+3)^2}{16}+\frac{(y-1)^2}{16}=1$

$\frac{(x+3)^2}{4}+\frac{(y-1)^2}{16}=1 \quad X=x+3 \quad Y=y-1$

$\frac{X^2}{4}+\frac{Y^2}{16}=1$ Major axis Y-axis

$a^2=16 \quad a=4 \quad b^2=4 \quad b=2$

$c^2=a^2-b^2=16-4=12 \Rightarrow c=\sqrt{12}=2\sqrt{3}$

Center $(-3,1)$

vertices $(h\pm a,k)$: $(-3,1\pm 4)=(-3,3)$; $(-3,-3)$

Foci $(h\pm c,k)$: $(-3,1\pm 2\sqrt{3})=(-3,1+2\sqrt{3})$; $(-3,1-2\sqrt{3})$

Length of major axis $2a=8$

Length of minor axis $2b=2(2)=4$

Length of latus rectum $=2\frac{b^2}{a}=2\frac{4}{4}=2$

Ex 5.2 - 8(v)

Identify type of conic and find center , foci , vertices and directrices of $18x^2+12y^2-144x+48y+120=0$

Solution :

$18x^2+12y^2-144x+48y+120=0$

$18x^2+12y^2-144x+48y=-120$

$18(x^2-8x)+12(y^2+4y)=-120$

$18(x^2-8x+16)+12(y^2+4y+4)$

$=-120+288+48=216$

$\div 216 \quad \frac{18(x-4)^2}{216}+\frac{12(y+2)^2}{216}=1$

$\frac{(x-4)^2}{12}+\frac{(y+2)^2}{18}=1 \quad X=x-4 \quad Y=y+2$

$\frac{X^2}{12}+\frac{Y^2}{18}=1$ Major axis parallel to y-axis ;

($a^2=18$, $a=\sqrt{18}=3\sqrt{2}$ & $b^2=12 \quad b=\sqrt{12}=2\sqrt{3}$)

$c=\sqrt{a^2-b^2}=\sqrt{18-12}=\sqrt{6}$

$\frac{a}{e}=\frac{a^2}{c}=\frac{18}{\sqrt{6}}=\frac{3.\sqrt{6}.\sqrt{6}}{\sqrt{6}}=3\sqrt{6}$

Center $(4,-2)$

Vertices $(h,k\pm a)=(4,-2\pm 3\sqrt{2})=(4,-2+3\sqrt{2});(4,-2-3\sqrt{2})$

Foci$(h,k\pm c)=(4,-2\pm\sqrt{6})=(4,-2+\sqrt{6});(4,-2-\sqrt{6})$

Eqn of directrices: $Y=\pm\frac{a}{e} \Rightarrow y+2=\pm 3\sqrt{6}$

i.e $y=-2+3\sqrt{6}$, $y=-2-3\sqrt{6}$

Find, eccentricity, center, vertices , foci of $36x^2+4y^2-72x+32y-44=0$

Solution :

$36x^2+4y^2-72x+32y-44=0$

$36x^2-72x+4y^2+32y=44$

$36(x^2-2x)+4(y^2+8y)=44$

$36(x^2-2x+1)+4(y^2+8y+16)=44+36+64=144$

$\div 225 \quad \frac{36(x-1)^2}{144}+4\frac{(y+4)^2}{144}=1$

$\frac{(x-1)^2}{4}+\frac{(y+4)^2}{36}=1 \quad X=x-1 \quad Y=y+4$

$\frac{X^2}{4}+\frac{Y^2}{36}=1$ Major axis parallel to Y-axis

$\{ a^2=36 \Rightarrow a=6 ;\ b^2=4 \Rightarrow b=2 \}$

$c^2=a^2-b^2=36-4=32$

$\Rightarrow c=\pm\sqrt{32}=\pm\sqrt{4x4x2}=\pm 4\sqrt{2}$

center $=(1,-4)$

vertices $(h,k\pm a)=(1,-4\pm 6)=(1,-4+6),(1,-4-6)$

$=(1,2),(1,-10)$

Foci $(h,k\pm c)=(1,-4\pm\sqrt{32})$

$=(1,-4+4\sqrt{2});(1,-4-4\sqrt{2})$

$e=\frac{c}{a}=\frac{4\sqrt{2}}{6}=\frac{2\sqrt{2}}{3}$

Example 5.24

Find the centre, foci and e of hyperbola

$\mathbf{11x^2 - 25y^2 - 44x + 50y - 256 = 0}$

Solution :

$11x^2 - 25y^2 - 44x + 50y - 256 = 0$

$11x^2 - 44x - 25y^2 + 50y = 256$

$11(x^2 - 4x) - 25(y^2 - 2y) = 256$

$11(x^2 - 4x + 4) - 25(y^2 - 2y + 1) = 256 + 44 - 25$

$11(x - 2)^2 - 25(y - 1)^2 = 275$

$\div 275 \quad \frac{11(x-2)^2}{275} - \frac{25(y-1)^2}{275} = 1$

$\Rightarrow \frac{(x-2)^2}{25} - \frac{(y-1)^2}{11} = 1 \quad X = x - 2 \quad Y = y - 1$

$\frac{x^2}{25} - \frac{y^2}{11} = 1$ Transverse axis parallel to x-axis

$a^2 = 25 \Rightarrow a = 5$ & $b^2 = 11 \quad b = \sqrt{11}$

$c^2 = a^2 + b^2 = 25 + 11 = 36 \Rightarrow c = \pm 6$

centre $= (2,1)$

Foci $(h \pm ae, k) = (2 \pm 6,1) = (2 + 6,1); (2 - 6,1)$

$= (8,1); (-4,1)$

$e = \frac{c}{a} = \frac{6}{5}$

Ex 5.4 (3)

Show that the line $\mathbf{x - y + 4 = 0}$ touches Ellipse $\mathbf{x^2 + 3y^2 = 12}$. Also find the co. ordinates of point of contact.

Solution:

$x - y + 4 = 0 \qquad x^2 + 3y^2 = 12$

$-y = -x - 4 \qquad \frac{x^2}{12} + \frac{3y^2}{12} = 1$

$\Rightarrow y = x + 4 \qquad \frac{x^2}{12} + \frac{y^2}{4} = 1$

$m = 1 \quad c = 4$

$a^2 = 12 \quad b^2 = 4$

condition: $c^2 = a^2m^2 + b^2$

L.H.S $\quad c^2 = 4^2 = 16$

R.H.S: $\quad a^2m^2 + b^2 = 12(1) + 4 = 12 + 4 = 16$

L.H.S=R.H.S

$\therefore$ line touches Ellipse

Point of contact $= \left(-\frac{a^2m}{c}, \frac{b^2}{c}\right)$

$= \left(-\frac{12(1)}{4}, \frac{4}{4}\right) = (-3,1)$

Exercise 5.2 - 8(vi)

Identify the conic and find centre , foci , vertices and directrices of $\mathbf{9x^2 - y^2 - 36x - 6y + 18 = 0}$

solution :

$9x^2 - y^2 - 36x - 6y + 18 = 0$

$9x^2 - 36x - y^2 - 6y = -18$

$9(x^2 - 4x) - (y^2 + 6y) = -18$

$9(x^2 - 4x + 4) - (y^2 + 6y + 9) = -18 + 36 - 9$

$9(x - 2)^2 - (y + 3)^2 = 9$

$\div 9 \quad \frac{(x-2)^2}{1} - \frac{(y+3)^2}{9} = 1 \quad X = x - 2 \quad Y = y + 3$

$\frac{X^2}{1} - \frac{Y^2}{9} = 1$ Transverse axis parallel to x - axis

$\{a^2 = 1 \Rightarrow a = 1 ; \ b^2 = 9 \Rightarrow b = 3\}$

$c^2 = a^2 + b^2 = 1 + 9 = 10 \Rightarrow c = \sqrt{10}$

centre $= (2, -3)$

Vertices $(h \pm a, k) = (2 \pm 1, -3) = (2 + 1, -3); (2 - 1, -3)$

$= (3, -3);(1, -3)$

Foci $(h \pm a,k) = (2 \pm \sqrt{10}, -3) = (2+\sqrt{10}, -3);(2-\sqrt{10}, -3)$

Eqn of directrices $X = \pm \frac{a}{e}$: $\left\{\frac{a}{e} = \frac{a^2}{c} = \frac{1}{\sqrt{10}}\right\}$

$x - 2 = \pm \frac{1}{\sqrt{10}} \Rightarrow x = 2 + \frac{1}{\sqrt{10}}$, $x = 2 - \frac{1}{\sqrt{10}}$

CREATED.

Prove that the line $\mathbf{5x + 12y = 9}$ touches $\mathbf{x^2 - 9y^2 = 9}$. Find point of contact.

Solution:

$5x + 12y = 9 \qquad x^2 - 9y^2 = 9$

$\Rightarrow 12y = -5x + 9 \qquad \frac{x^2}{9} - \frac{y^2}{1} = 1$

$\Rightarrow y = \frac{-5}{12}x + \frac{9}{12} \qquad a^2 = 9 \quad b^2 = 1$

$\Rightarrow y = \frac{-5}{12}x + \frac{3}{4} \qquad m = -\frac{5}{12} \quad c = \frac{3}{4}$

Condition: $c^2 = a^2m^2 - b^2$

L.H.S: $c^2 = \left(\frac{3}{4}\right)^2 = \frac{9}{16}$

R.H.S: $a^2m^2 - b^2 = 9\left(\frac{-5}{12}\right)^2 - 1 = 9\left(\frac{25}{144}\right) - 1$

$= \frac{225-144}{144} = \frac{81}{144} = \frac{9}{16}$

LHS=RHS Line touch hyperbola

pt of contact $= \left(-\frac{a^2m}{c}, -\frac{b^2}{c}\right)$

$= \left(\frac{-9\left(-\frac{5}{12}\right)}{\frac{3}{4}}, \frac{-1}{\frac{3}{4}}\right) = \left(5, -\frac{4}{3}\right)$

Exercise 5.5.(1)

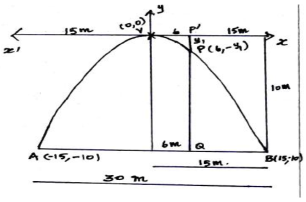
A bridge has a parabolic arch that is $10\ \mathbf{m}$ high in the centre and $30\ \mathbf{m}$ wide at the bottom. Find the height of the arch $6\ \mathbf{m}$ from the centre, on either sides.

solution:

Vertex (0,0) Parabola open downward

Equation $x^2 = -4ay$

Pt $B(15,-10)$ lies on parabola



$\therefore\ (15)^2 = -4a(-10)$

$\Rightarrow\ 225 = 4a(10)\ \Rightarrow\ 4a = \frac{225}{10}$

$\therefore$ EQN $\ x^2 = -\frac{225}{10}\ y$

$\Rightarrow\ x^2 = -\frac{45}{2}\ y$

PQ is the height of arch 6 m to the right from center.

$PP' = y_1$

$\therefore P(6,-y_1)$ lies on parabola : $x^2 = -\frac{45}{2}y$

$$\Rightarrow\ 6^2 = -\frac{45}{2}(-y_1)$$

$$\Rightarrow\ y_1 = \frac{2\times 36}{45} = \frac{8}{5}$$

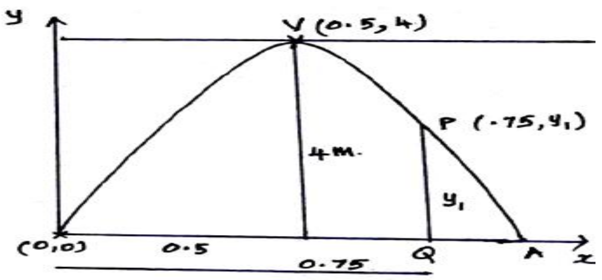
$$y_1 = 1.6\ \text{m}$$

$\therefore$ Height of arch $= 10 - y_1 = 10 - 1.6 = 8.4$ m

Exercise 5.5.(3)

At a water fountain, water attains a maximum height of $4\ \mathbf{m}$ at horizontal distance of $0.5\ \mathbf{m}$ from its origin. If the path of water is a parabola, find the height of water at a horizontal distance of $0.75\ \mathbf{m}$ from the point of origin.

SOLUTION:



From the diagram $V(0.5,4) = (h,k)$

Parabola open downward.

Equation: $(x-h)^2 = -4a(y-k)\ \Rightarrow (x-0.5)^2 = -4a(y-4$

(0,0) lies on parabola

$(0-0.5)^2 = -4a(0-4) \Rightarrow (-0.5)^2 = 4a(4) \Rightarrow 4a = \frac{0.25}{4}$

$\therefore$ Eqn: $(x-0.5)^2 = -\frac{0.25}{4}(y-4)$

Let $OQ = 0.75$

$PQ = y_1$;

$\therefore P(0.75, y_1)$ lies on parabola.

$(x-0.5)^2 = -\dfrac{0.25}{4}(y-4)$

$(0.75-0.5)^2 = \frac{-0.25}{4}(y_1-4)$

$\Rightarrow\ (0.25)^2 = -\frac{0.25}{4}(y_1-4)$

$y_1 - 4 = \frac{-4\times(0.25)^2}{0.25}$

$y_1 - 4 = -4\times 0.25 = -1$

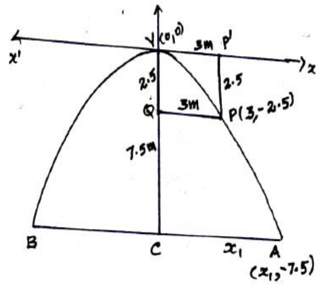
Height of water $= y_1 = -1 + 4 = 3$m .

Exercise 5.5.(8)

4) Assume that water issuing from the end of a horizontal pipe, $7.5\ \mathbf{m}$ above the ground, describes a parabolic path. The vertex of the parabolic path is at the end of the pipe . At a position 2.5m below the outward $3\ \mathbf{m}$ beyond the vertical line through the end of the pipe. How far beyond this vertical line will the water strike the ground?

SOLUTION:

From the diagram, vertex V(0,0)



Parabola open downward.

Equation $x^2 = -4ay \quad -(1)$

Let $VP' = 3$ m , $VQ = 2.5$

$\therefore P(3,-2.5)$ lies on parabola.

sub in (1) $3^2 = -4a(-2.5)$

$9 = 4a(2.5)$

$\Rightarrow 4a = \frac{9}{2.5}$

$\therefore$ (1) becomes $x^2 = -\left(\frac{9}{2.5}\right)y$

Let $AC = x_1$ be the distance.

$\therefore A(x_1,-7.5)$ lies on parabola

$\therefore\ x_1^2 = -\left(\frac{9}{2.5}\right)(-7.5)$

$\Rightarrow x_1^2 = -9(-3)$

$x_1^2 = 27$

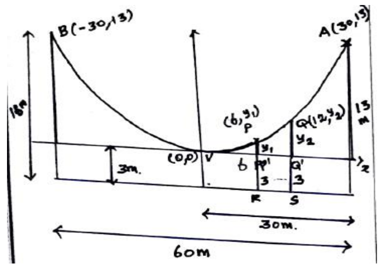
$\Rightarrow\ x_1 = \sqrt{27} = 3\sqrt{3}$ m

Exercise 5.5 (5)

Parabolic cable of a $60\ \mathbf{m}$ portion of the roadbed of a suspension bridge are positioned as shown below. Vertical Cables are to be spaced every 6m along this portion of the roadbed. Calculate the lengths of first two of these vertical cables from the vertex.

Solution :

From the Diagram V(0,3)



Parabola open upward

Equation $x^2 = 4ay$

A (30,13) lies on parabola

$30^2 = 4a(13)$

$\Rightarrow 4a = \frac{900}{13}$

$\therefore$ Eqn $x^2 = \frac{900}{13}y$

$VP' = 6,\ PP' = y_1\ \ \therefore P(6,y_1)$

$(6,y_1)$ lies on the parabola $6^2 = \frac{900}{13}y_1$

$y_1 = \frac{36\times 13}{900} = \frac{4\times 13}{100} = \frac{52}{100} = 0.52$

$\therefore$ Height of cable $PR = 0.52 + 3 = 3.52$m

$VQ' = 12 \quad QQ' = y_2$

$(12,y_2)$ lies on parabola

$12^2 = \frac{900}{13}y_1$

$y_1 = \frac{144\times 13}{900} = \frac{16\times 13}{100} = \frac{208}{900} = 2.08$

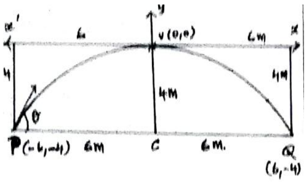
$\therefore$ Height of cable is $= 3 + 2.08 = 5.08$m

9) On lighting a rocket cracker if gets projected in a parabolic path and reaches a maximum height of 4m when it is 6m away from the point of projection Finally is reaches the ground 12 m away from the starting point . Find the angle of projection at P.

Solution :

From the diagram vertex V(0,0)

Parabola open downward



$\therefore\ x^2 = -4ay \quad -(1)$

$PC = 6\text{ m} \quad VC = 4\text{ m}$

$\therefore$ Pt $P(-6, -4)$ lies on parabola

$\therefore (-6)^2 = -4a(-4)$

$\Rightarrow\ 36 = 4a(4)$

$4a = \frac{36}{4} = 9$

$\Rightarrow$ - Becomes $x^2 = -9y$ -(2)

Let θ be the angle of projection.

To find θ, Differentiate (2) with respect to x

$2x = -9\frac{dy}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-2x}{9}$

$\Rightarrow m = \tan\theta = \frac{dy}{dx}$ at $(-6, -4) \Rightarrow \tan\theta = \frac{-2(-6)}{9} = \frac{12}{9} = \frac{4}{3}$

$\therefore$ Angle of projection at p $\quad \theta = \tan^{-1}\left(\frac{4}{3}\right)$

---

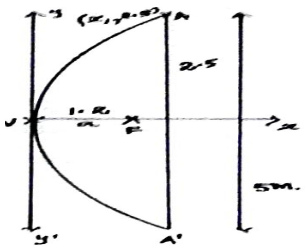
4) An engineer design a satellite dish with a parabolic cross section .The dish is 5m wide at the opening , and the focus is placed 12 m from The vertex.

(i) Position a coordinate system with the origin at the vertex and the x-axis on the parabola's axis of symmetry and find an equation of the parabola .

(ii) Find the depth of the satellite dish at the vertex

SOLUTION :

From the diagram



$V(0, \theta)$

Parabola open right ward

$y^2 = 4ax$

given $a = 1.2$

$\therefore y^2 = 4(1.2)x$

$y^2 = 4.8x$

let x_1 be depth $\Rightarrow A\ (x_1, 2.5)$ lies on parabola

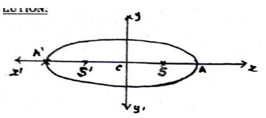
$(2.5)^2 = 4.8x_1 \Rightarrow \quad$ Depth $\ x_1 = \frac{6.25}{4.8} = 1.3$ m.

EXAMPLE 5.32: The maximum and minimum distances of the Earth from the Sun respectively are 152×10^6 km and 94.5×10^6 km . The Sun is at one focus of the elliptical orbit. Find the distance from the Sun to the other focus.

SOLUTION:

Shortest distance $= SA = 94.5 \times 10^6$

$\Rightarrow CA - CS = 94.5 \times 10^6$



$\Rightarrow a - ae = 94.5 \times 10^6$

Longest distance $= SA' = 152 \times 10^6$

$\Rightarrow CA' + CS = 152 \times 10^6$

$\Rightarrow a + ae = 152 \times 10^6$

$a + ae = 152 \times 10^6$

$a - ae = 94.5 \times 10^6$

$2ae = 57.5 \times 10^6 \qquad \Rightarrow 2ae = 575 \times 10^5$Km

Distance of sun from other focus 575×10^5km

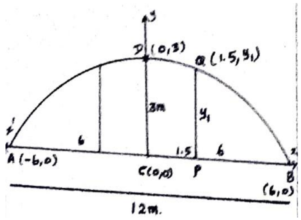
EXAMPLE 5.31: A semi elliptical archway over one way road way has and Height of 3m and width of 2m . The truch has a width of 3 m and a height of 2. 7 m . Will the truck clear the opening of the archway

Solution :

Archway is in the form of Semi ellipse.

center(0,0)

Eqn : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad$ - (1)



Given: $AB = 2a = 12 \quad \Rightarrow a = b$

$CD = b = 3$

$\therefore$ (1) becomes

$\frac{x^2}{6^2} + \frac{y^2}{3^2} = 1 \Rightarrow \frac{x^2}{36} + \frac{y^2}{9} = 1$

Let y_1 be the height of arch 1.5m - to the right from the center.

i.e Q $(1.5, y_1)$ lies on ellipse

$\Rightarrow \quad \frac{(1.5)^2}{36} + \frac{y_1^2}{9} = 1 \ \Rightarrow\ \frac{y_1^2}{9} = 1 - \frac{2.25}{36}$

$\frac{y_1^2}{9} = \frac{36 - 2.25}{36}$

$\frac{y_1^2}{9} = \frac{33.75}{36} \ \Rightarrow \quad y_1^2 = \frac{33.75}{36} \times 9 = \frac{33.75}{4}$

$y_1^2 = 8.43 \quad \Rightarrow\ y_1 = 2.9$m

$\because$ Heignt of truck is $2.7 < 2.9$ m

Truck will clear the opening of archway

---

6) Cross section of nuclear cooling tower is in the shape of a hyperbola with equation $\frac{x^2}{30^2} - \frac{y^2}{44^2} = 1$, tower is 150 m tall and the distance from the top of the tower to the centre of the hyperbola is half the distance from the base of the tower to the centre of the hyperbola . Find the diameter of the top and base of the tower

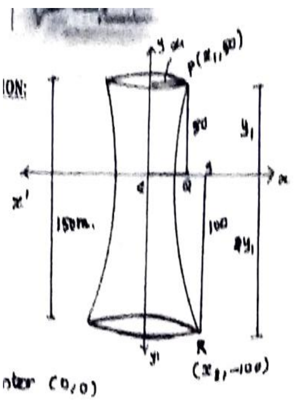
Solution :

Center (0,0) $\quad$ Equation of Hyperbola $\frac{x^2}{30^2} - \frac{y^2}{44^2} = 1$

Top of tower from center $= y_1$

bottom of tower from center $= 2y_1$

$y_1 + 2y_1 = 150 \ \Rightarrow\ 3y_1 = 150$



$y_1 = \frac{150}{3} = 50$ m

let x_1 be the radius of top of tower

$\therefore$ P$(x_1, 50)$ lies on Hyperbola .

$\frac{x_1^2}{30^2} - \frac{50^2}{44^2} = 1$

$\frac{x_1^2}{30^2} = 1 + \frac{2500}{1936}$

$= \frac{1936+2500}{1936} = \frac{4456}{1936}$

$x_1^2 = \frac{30^2}{44^2}(4436)$

$x_1 = \frac{30}{44}\sqrt{4436} = 45.41$

diameter $2x_1 = 2(45.41) = 90.82$

Let x_2 be the radius of bottom.

R$(x_2, -100)$ lies on hyperbola $\frac{x_2^2}{30^2} - \frac{100^2}{44^2} = 1$

$\frac{x_2^2}{30^2} = 1 + \frac{10000}{1936} = \frac{1936+10000}{1936} = \frac{11936}{1936}$

$x_2^2 = \frac{30^2}{44^2}(11936)$

$x_2 = \frac{30}{44}\sqrt{11936}$

Diameter $= 2x_2 = 148.98$m

Ex 5.5 Q.no (2)

A turnel through a mountain for o four lane highway is to have a elliptical opening . The total width of the highwoy (not the opening) is to be 16 m , and the height at the edge of the road must be sufficient for a truck 4m high to clear if the highest point of the opening is to be 5 m approximately. How wide must the opening be ?

Solution :

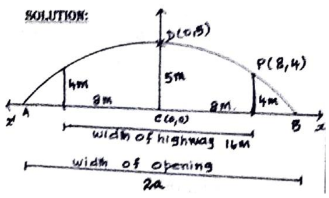
Opening of the tunnel is in elliptical shape.

Let mid pt of base be center C(0,0)

$AB = 2a =$ widh of opening

$AC = CB = a$

height $= b = 5$



Eqn of ellipse : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\frac{x^2}{a^2} + \frac{y^2}{5^2} = 1 \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{25} = 1$

width of highway $= 16$ m

At the edge, height is sufficient to clear a truck of 4 m height

$\therefore$ P(8,4) lies on ellipse

$\frac{8^2}{a^2} + \frac{4^2}{25} = 1 \Rightarrow \quad \frac{8^2}{a^2} = 1 - \frac{16}{25} = \frac{25-16}{25} = \frac{9}{25}$

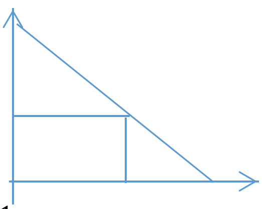
$a^2 = \frac{8^2 \times 25}{9} = \frac{8^2 \times 5^2}{3^2} \Rightarrow \quad a = \frac{8 \times 5}{3} = \frac{40}{3} = 13.33 \ (\because a > 0)$

width of opening $= 2a = 2 \times 13.33 = 26.66 \approx 26.7$ m

---

7) A rod of length 1.2 m moves with its ends always touching the coordinate axes. The locus of a point P on the rod, which is 0.3 m from the end in contact with x-oxis is an ellipse. Find the eccentricity.

SOLUTION:



$AB = 1.2 \quad AP = 0.3$

$BP = 1.2 - 0.3 = 0.9$

Let θ be the angle made with x-axis

Eqn : $\cos^2\theta + \sin^2\theta = 1$

$\frac{x_1^2}{(0.9)^2} + \frac{y_1^2}{(0.3)^2} = 1$ i.e $\frac{x_1^2}{0.8} + \frac{y_1^2}{0.09} = 1$

$e = \sqrt{\frac{a^2 - 1^2}{a^2}} = \sqrt{\frac{0.81 - 0.09}{0.81}} = \sqrt{\frac{0.72}{0.81}} = \sqrt{\frac{72}{81}}$

$e = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}$.

EXAMPLE 5.40: Two coast guard stations are located 600 km apart at points A(0, 0) and B(0, 600) A distres signal from a ship at P is received af slightly different times by two stations . If determined that the ship is 200 km farther from station A than it is from station B Determine the equation of hyperbola that passes through the location of the ship .

SOLUTION:

A (0,0) (0,600) Foci

Center $= \left(\frac{0+0}{2}, \frac{0+600}{2}\right) = (0,300)$

Transverse axis y-axis Eqn : $\frac{(y-300)^2}{a^2} - \frac{(x)^2}{b^2} = 1$

Given AB= $2ae = 600 \Rightarrow ae = 300$

$|AP - BP| = 2a = 200 \Rightarrow a = 100$

$b^2 = (ae)^2 - a^2 = 300^2 - 100^2$

$= 90000 - 10000 = 80000$

$\therefore$ Equation $\frac{(y-300)^2}{10000} - \frac{x^2}{80000} = 1$.

---

10) Points A and B are 10 km apart and it is determine from the sound of an explosion heard at those points at different times that the location of the explosion is 6 km closer to A than B . Show that the location of the explosion is restricted to a particular curve and find an equation of if

SOLUTION:

Let A and B be the focus.

$AB = 2ae = 10 \Rightarrow ae = 5$

Let p be the point of explosion.

$|AP - BP| = 2a = 6$

$\therefore b^2 = (ae)^2 - a^2 = 5^2 - 3^2 = 25 - 9 = 16$

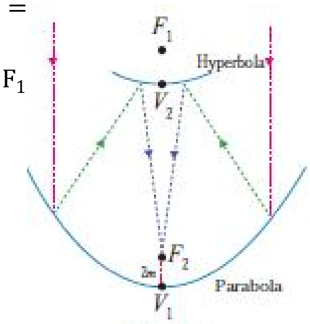
Locus of pt p is Hyperbola center (0,0)

$\frac{x^2}{9} - \frac{y^2}{16} = 1$

EXAMPLE 5.35: Certain telescopes contain both parabolic mirror and a hyperbolic mirror. In the telescope shown in figure the parabola and hyperbola share focus F_1 which is 14 m above the vertex of the parabola. The hyperbola's second focus F_2 is 2m above the parabola's vertex . the vertex of the hyperbolic mirror is 1 m below F_1 . Position of coordinate system with the origin af the centre of the hyperbola and with the foci on the y -axis Then find the equation of the hyperbola.

Solution :

$V_1 =$ vertex of parabola & $V_2 =$ Vertex of hyperbola



F_1& F_2 are Foci of Hyperbola but F_1 is focus of parabola also

$V_1F_1 = 14$ m $\quad V_1F_2 = 2$m

$F_1F_2 = 2ae = 14 - 2 = 12$m

$CF_1 = ae = 6$m

$a = 6 - 1 = 5$m $\Rightarrow a^2 = 25$

$b^2 = (ae)^2 - a^2 = 6^2 - 5^2$

$b^2 = 36 - 25 = 11$

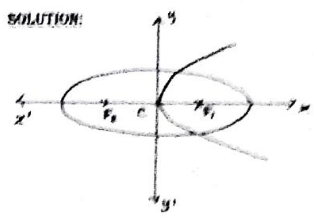
Transverse axis y axis center (0,0).

$\therefore \frac{y^2}{25} - \frac{x^2}{11} = 1$

EXAMPLE 5.36: An equation of the elliptical part of an optical lens system is $\frac{x^2}{16} + \frac{y^2}{9} = 1$. The parabolic part of the system has a focus in common with the right focus of the ellipse .The vertex of the parabola is at the origin and the parabola opens to the right. Determine the equation of the parabola.

Solution :

$\frac{x^2}{16} + \frac{y^2}{9} \quad a^2 = 16$



$b^2 = 9ae = \sqrt{a^2 - b^2}$

$= \sqrt{16 - 9}$

Foci of ellipse are

$(\sqrt{7}, 0), (-\sqrt{7}, 0)$

Given parabolic part of focus

coincides with right focus of ellipse parabola opens right.

$\therefore$ Eqn is $y^2 = 4ax$

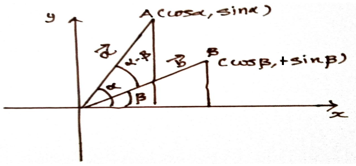
$\therefore y^2 = 4\sqrt{7}x$

CHAPTER 6 - VECTOR ALGEBRA
(5 MARKS ONLY)

EXERCISE 6.1 (5): Prove by vector method:

$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

Solution:



$|\vec{a}| = |\vec{b}| = 1$

$\angle AOB = \alpha - \beta$

$A(\cos\alpha, \sin\alpha)$

$B(\cos\beta, \sin\beta)$

$\vec{a} = \cos\alpha\,\hat{i} + \sin\alpha\,\hat{j}$ & $\vec{b} = \cos\beta\,\hat{i} + \sin\beta\,\hat{j}$

$\vec{b}.\vec{a} = |\vec{b}||\vec{a}|\cos(\alpha - \beta) = (1)(1)\cos(\alpha - \beta)$

$= \cos(\alpha - \beta)$ ____(1)

$\vec{b}.\vec{a} = (\cos\beta\,\hat{i} + \sin\beta\,\hat{j}).(\cos\alpha\,\hat{i} + \sin\alpha\,\hat{j})$

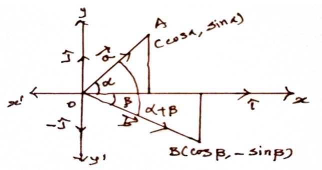
$= \cos\alpha\cos\beta + \sin\alpha\sin\beta$ _____(2)

From (1) and (2) $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

EXAMPLE 6.3 : Prove by vector method:

$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ Solution:

Solution:



$|\vec{a}| = |\vec{b}| = 1$ &

$\angle AOB = \alpha + \beta$

$A(\cos\alpha, \sin\alpha)$ &

$B(\cos\beta, -\sin\beta)$

$\vec{a} = \cos\alpha\,\hat{i} + \sin\alpha\,\hat{j}$ & $\vec{b} = \cos\beta\,\hat{i} - \sin\beta\,\hat{j}$

$\vec{b}.\vec{a} = |\vec{b}||\vec{a}|\cos(\alpha + \beta) = (1)(1)\cos(\alpha + \beta)$

$= \cos(\alpha + \beta)$ ____(1)

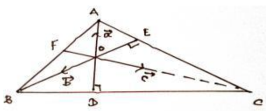
$\vec{b}.\vec{a} = (\cos\beta\,\hat{i} - \sin\beta\,\hat{j}).(\cos\alpha\,\hat{i} + \sin\alpha\,\hat{j})$

$= \cos\alpha\cos\beta - \sin\alpha\sin\beta$ _____(2)

From (1) and (2) $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

EXAMPLE 6.7 : Prove by vector method that the perpendiculars (ALTITUDES) drawn from the vertices to the opposite sides of a triangle are concurrent.

Solution:



In triangle ABC,

Altitudes AD, BE meet at O.

To prove the third altitude from c to AB also pass through O.

$AD \perp BC \Rightarrow OA \perp BC$	$BE \perp CA \Rightarrow OB \perp CA$
$\Rightarrow \overrightarrow{OA} \perp \overrightarrow{BC}$	$\Rightarrow \overrightarrow{OB} \perp \overrightarrow{CA}$
$\Rightarrow \overrightarrow{OA}\,.\,\overrightarrow{BC} = 0$	$\Rightarrow \overrightarrow{OB}\,.\,\overrightarrow{CA} = 0$
$\Rightarrow \overrightarrow{OA}.(\overrightarrow{OC} - \overrightarrow{OB}) = 0$	$\Rightarrow \overrightarrow{OB}.(\overrightarrow{OA} - \overrightarrow{OC}) = 0$
$\Rightarrow \overrightarrow{OA}.\overrightarrow{OC} - \overrightarrow{OA}.\overrightarrow{OB} = 0$	$\Rightarrow \overrightarrow{OB}.\overrightarrow{OA} - \overrightarrow{OB}.\overrightarrow{OC} = 0$
$\Rightarrow \overrightarrow{OA}.\overrightarrow{OC} = \overrightarrow{OA}.\overrightarrow{OB}$	$\Rightarrow \overrightarrow{OB}.\overrightarrow{OA} = \overrightarrow{OB}.\overrightarrow{OC}$

$\overrightarrow{OA}.\overrightarrow{OC} = \overrightarrow{OA}.\overrightarrow{OB}$ & $\overrightarrow{OB}.\overrightarrow{OA} = \overrightarrow{OB}.\overrightarrow{OC}$

$\Rightarrow \overrightarrow{OA}.\overrightarrow{OC} = \overrightarrow{OB}.\overrightarrow{OC}$

$\Rightarrow \overrightarrow{OC}.\overrightarrow{OA} = \overrightarrow{OC}.\overrightarrow{OB}$

$\Rightarrow \overrightarrow{OC}.\overrightarrow{OB} - \overrightarrow{OC}.\overrightarrow{OA} = 0$

$\Rightarrow \overrightarrow{OC}.(\overrightarrow{OB} - \overrightarrow{OA}) = 0 \quad \Rightarrow \overrightarrow{OC}.\overrightarrow{AB} = 0$

$\Rightarrow \overrightarrow{OC} \perp \overrightarrow{AB}$

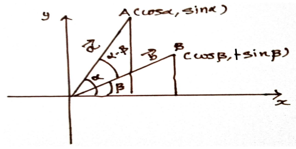
$\Rightarrow OC \perp AB$

Altitude from C to AB also pass through O

EXAMPLE 6.5: Prove by vector method:

$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

Solution:



$|\vec{a}| = |\vec{b}| = 1$

$\angle AOB = \alpha - \beta$

$A(\cos\alpha, \sin\alpha)$

$B(\cos\beta, \sin\beta)$

$\vec{a} = \cos\alpha\,\hat{i} + \sin\alpha\,\hat{j}$ & $\vec{b} = \cos\beta\,\hat{i} + \sin\beta\,\hat{j}$

$\vec{b} \times \vec{a} = |\vec{b}||\vec{a}|\sin(\alpha - \beta)\,\hat{k} = (1)(1)\sin(\alpha - \beta)\,\hat{k}$

$= \sin(\alpha - \beta)\,\hat{k}$ _____(1)

$$\vec{b} \times \vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\beta & \sin\beta & 0 \\ \cos\alpha & \sin\alpha & 0 \end{vmatrix}$$

$= \hat{i}(0) - \hat{j}(0) + \hat{k}(\cos\beta\sin\alpha - \cos\alpha\sin\beta)$

$= \hat{k}(\cos\beta\sin\alpha - \cos\alpha\sin\beta)$ ______(2)

From (1) and (2)

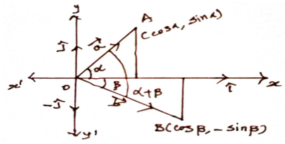
$\sin(\alpha - \beta)\,\hat{k} = \hat{k}(\cos\beta\sin\alpha - \cos\alpha\sin\beta)$

$\sin(\alpha - \beta) = (\cos\beta\sin\alpha - \cos\alpha\sin\beta)$

EXERCISE 6.1(10): Prove by vector method:-

$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

Solution:



$|\vec{a}| = |\vec{b}| = 1$

$\angle AOB = \alpha - \beta$

$A(\cos\alpha, \sin\alpha)$ & $B(\cos\beta, -\sin\beta)$

$\vec{a} = \cos\alpha\,\hat{i} + \sin\alpha\,\hat{j}$ & $\vec{b} = \cos\beta\,\hat{i} - \sin\beta\,\hat{j}$

$\vec{b} \times \vec{a} = |\vec{b}||\vec{a}|\sin(\alpha + \beta)\,\hat{k} = (1)(1)\sin(\alpha + \beta)\,\hat{k}$

$= \sin(\alpha + \beta)\,\hat{k}$ _____(1)

$$\vec{b} \times \vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\beta & -\sin\beta & 0 \\ \cos\alpha & \sin\alpha & 0 \end{vmatrix}$$

$= \hat{i}(0) - \hat{j}(0) + \hat{k}(\cos\beta\sin\alpha + \cos\alpha\sin\beta)$

$= \hat{k}(\cos\beta\sin\alpha + \cos\alpha\sin\beta)$ ______(2)

From (1) and (2)

$\sin(\alpha + \beta)\,\hat{k} = \hat{k}(\cos\beta\sin\alpha + \cos\alpha\sin\beta)$

$\sin(\alpha + \beta) = (\cos\beta\sin\alpha + \cos\alpha\sin\beta)$

Example 6.6: If D is the midpoint of the side BC of a triangle ABC, show by vector method that

$$|\overrightarrow{\mathbf{AB}}|^2 + |\overrightarrow{\mathbf{AC}}|^2 = 2\left(|\overrightarrow{\mathbf{AD}}|^2 + |\overrightarrow{\mathbf{BD}}|^2\right)$$

Solution:

In triangle ABC , D is mid point of BC

$BD = DC$ & $\overrightarrow{DB} = -\overrightarrow{DC}$

Equal magnitude but opposite direction

$|\overrightarrow{AB}|^2 + |\overrightarrow{AC}|^2$

$= |\overrightarrow{AB} + \overrightarrow{DB}|^2 + |\overrightarrow{AB} + \overrightarrow{DC}|^2 \qquad \overrightarrow{DC} = -\overrightarrow{DB}$

$= |\overrightarrow{AD} + \overrightarrow{DB}|^2 + |\overrightarrow{AD} - \overrightarrow{DB}|^2$

$= |\overrightarrow{AD}|^2 + |\overrightarrow{DB}|^2 + 2\overrightarrow{AD}\cdot\overrightarrow{DB} + |\overrightarrow{AD}|^2 + |\overrightarrow{DB}|^2 - 2\overrightarrow{AD}\cdot\overrightarrow{DB}$

$= 2|\overrightarrow{AD}|^2 + 2|\overrightarrow{DB}|^2$

$= 2\left(|\overrightarrow{AD}|^2 + |\overrightarrow{DB}|^2\right)$

$= 2\left(|\overrightarrow{AD}|^2 + |\overrightarrow{BD}|^2\right)$

Exercise 6.3 (4):

If $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}, \vec{b} = 3\hat{i} + 5\hat{j} + 2\hat{k}, \vec{c} = -\hat{i} - 2\hat{j} + 3\hat{k}$, verify that

(i) $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a}.\vec{c})\vec{b} - (\vec{b}.\vec{c})\vec{a}$

(ii) $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b}) \times \vec{c}$

Solution:

$\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}, \vec{b} = 3\hat{i} + 5\hat{j} + 2\hat{k}, \vec{c} = -\hat{i} - 2\hat{j} + 3\hat{k}$

(i) $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a}.\vec{c})\vec{b} - (\vec{b}.\vec{c})\vec{a}$

L.H.S

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & -1 \\ 3 & 5 & 2 \end{vmatrix} = \hat{i}(6+5) - \hat{j}(4+3) + \hat{k}(10-9)$$

$$= \hat{i}(11) - \hat{j}(7) + \hat{k}(1) = 11\hat{i} - 7\hat{j} + \hat{k}$$

$$(\vec{a} \times \vec{b}) \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 11 & -7 & 1 \\ -1 & -2 & 3 \end{vmatrix}$$

$$= \hat{i}(-21+2) - \hat{j}(33+1) + \hat{k}(-22-7)$$

$$= \hat{i}(-19) - \hat{j}(34) + \hat{k}(-29) = -19\hat{i} - 34\hat{j} - 29\hat{k}.$$

R.H.S. $\vec{a}.\vec{c} = (2\hat{i} + 3\hat{j} - \hat{k}).(-\hat{i} - 2\hat{j} + 3\hat{k})$

$$= 2(-1) + 3(-2) + (-1)(3) = -2-6-3 = -11$$

$\vec{b}.\vec{c} = (3\hat{i} + 5\hat{j} + 2\hat{k}).(-\hat{i} - 2\hat{j} + 3\hat{k})$

$$= 3(-1) + 5(-2) + 2(3) = -3-10+6 = -13+6 = -7$$

$(\vec{a}.\vec{c})\vec{b} - (\vec{b}.\vec{c})\vec{a}$

$= -11(3\hat{i} + 5\hat{j} + 2\hat{k}) - (-7)(2\hat{i} + 3\hat{j} - \hat{k})$

$= -33\hat{i} - 55\hat{j} - 22\hat{k} + 14\hat{i} + 21\hat{j} - 7\hat{k} = -19\hat{i} - 34\hat{j} - 29\hat{k}$

L. H.S = R.H.S $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a}.\vec{c})\vec{b} - (\vec{b}.\vec{c})\vec{a}$

(ii) $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$

L.H.S

$$\vec{b} \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 5 & 2 \\ -1 & -2 & 3 \end{vmatrix} = \hat{i}(15+4) - \hat{j}(9+2) + \hat{k}(-6+5)$$

$$= \hat{i}(19) - \hat{j}(11) + \hat{k}(-1) = 19\hat{i} - 11\hat{j} - 1\hat{k}.$$

$$\vec{a} \times (\vec{b} \times \vec{c}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & -1 \\ 19 & -11 & -1 \end{vmatrix}$$

$$= \hat{i}(-3-11) - \hat{j}(-2+19) + \hat{k}(-22-57)$$

$$= \hat{i}(-14) - \hat{j}(17) + \hat{k}(-79) = -14\hat{i} - 17\hat{j} - 79\hat{k}.$$

$\vec{a}.\vec{c} = (2\hat{i} + 3\hat{j} - \hat{k}).(-\hat{i} - 2\hat{j} + 3\hat{k})$

$$= 2(-1) + 3(-2) + (-1)(3) = -2-6-3 = -11$$

$\vec{a}.\vec{b} = (2\hat{i} + 3\hat{j} - \hat{k}).(3\hat{i} + 5\hat{j} + 2\hat{k})$

$$= 2(3) + 3(5) + (-1)(2) = 6+15-2 = 19$$

$(\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$

$= (-11)(3\hat{i} + 5\hat{j} + 2\hat{k}) - 19(-\hat{i} - 2\hat{j} + 3\hat{k})$

$= -33\hat{i} - 55\hat{j} - 22\hat{k} + 19\hat{i} + 38\hat{j} - 57\hat{k} = -14\hat{i} - 17\hat{j} - 79\hat{k}$

$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$

Example 6.23:

If $\vec{a} = \hat{i} - \hat{j}, \vec{b} = \hat{i} - \hat{j} - 4\hat{k}, \vec{c} = 3\hat{j} - \hat{k}, \vec{d} = 2\hat{i} + 5\hat{j} + \hat{k}$, verify that

(i) $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a},\vec{b},\vec{d}]\vec{c} - [\vec{a},\vec{b},\vec{c}]\vec{d}$

Solution:

$\vec{a} = \hat{i} - \hat{j}, \vec{b} = \hat{i} - \hat{j} - 4\hat{k}, \vec{c} = 3\hat{j} - \hat{k}, \vec{d} = 2\hat{i} + 5\hat{j} + \hat{k}$

(i) $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a},\vec{b},\vec{d}]\vec{c} - [\vec{a},\vec{b},\vec{c}]\vec{d}$

L.H.S

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 0 \\ 1 & -1 & -4 \end{vmatrix} \begin{aligned} &= \hat{i}(4-0) - \hat{j}(-4-0) + \hat{k}(-1+1) \\ &= \hat{i}(4) - \hat{j}(-4) + \hat{k}(0) = 4\hat{i} + 4\hat{j} \end{aligned}$$

$$\vec{c} \times \vec{d} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 3 & -1 \\ 2 & 5 & 1 \end{vmatrix} \begin{aligned} &= \hat{i}(3+5) - \hat{j}(0+2) + \hat{k}(0-6) \\ &= \hat{i}(8) - \hat{j}(2) + \hat{k}(-6) = 8\hat{i} - 2\hat{j} - 6\hat{k} \end{aligned}$$

$$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 4 & 0 \\ 8 & -2 & -6 \end{vmatrix}$$

$$= \hat{i}(-24-0) - \hat{j}(-24-0) + \hat{k}(-8-32)$$

$$= \hat{i}(-24) - \hat{j}(-24) + \hat{k}(-40) = -24\hat{i} + 24\hat{j} - 40\hat{k}$$

R.H.S

$$[\vec{a},\vec{b},\vec{d}] = \begin{vmatrix} 1 & -1 & 0 \\ 1 & -1 & -4 \\ 2 & 5 & 1 \end{vmatrix} = 1(-1+20) + 1(1+8) + 0(5+2)$$

$$= 1(19) + 1(9) + 0 = 19+9 = 28$$

$$[\vec{a},\vec{b},\vec{c}] = \begin{vmatrix} 1 & -1 & 0 \\ 1 & -1 & -4 \\ 0 & 3 & -1 \end{vmatrix} = 1(1+12) + 1(-1-0) + 0(3+0)$$

$$= 1(13) + 1(-1) + 0 = 13-1 = 12$$

$[\vec{a},\vec{b},\vec{d}]\vec{c} - [\vec{a},\vec{b},\vec{c}]\vec{d} = 28(3\hat{j} - \hat{k}) - 12(2\hat{i} + 5\hat{j} + \hat{k})$

$$= 84\hat{j} - 28\hat{k} - 24\hat{i} - 60\hat{j} - 12\hat{k} = -24\hat{i} + 24\hat{j} - 40\hat{k}$$

L.H.S=R.H.S $\underline{(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a},\vec{b},\vec{d}]\vec{c} - [\vec{a},\vec{b},\vec{c}]\vec{d}}$

(ii) $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a},\vec{c},\vec{d}]\vec{b} - [\vec{b},\vec{c},\vec{d}]\vec{a}$

L.H.S:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 0 \\ 1 & -1 & -4 \end{vmatrix} \begin{aligned} &= \hat{i}(4-0) - \hat{j}(-4-0) + \hat{k}(-1+1) \\ &= \hat{i}(4) - \hat{j}(-4) + \hat{k}(0) = 4\hat{i} + 4\hat{j} \end{aligned}$$

$$\vec{c} \times \vec{d} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 3 & -1 \\ 2 & 5 & 1 \end{vmatrix} \begin{aligned} &= \hat{i}(3+5) - \hat{j}(0+2) + \hat{k}(0-6) \\ &= \hat{i}(8) - \hat{j}(2) + \hat{k}(-6) = 8\hat{i} - 2\hat{j} - 6\hat{k} \end{aligned}$$

$$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 4 & 0 \\ 8 & -2 & -6 \end{vmatrix}$$

$$= \hat{i}(-24-0) - \hat{j}(-24-0) + \hat{k}(-8-32)$$

$$= \hat{i}(-24) - \hat{j}(-24) + \hat{k}(-40) = -24\hat{i} + 24\hat{j} - 40\hat{k}$$

$$[\vec{a},\vec{c},\vec{d}] = \begin{vmatrix} 1 & -1 & 0 \\ 0 & 3 & -1 \\ 2 & 5 & 1 \end{vmatrix} \begin{aligned} &= 1(3+5) + 1(0+2) + 0(0-6) \\ &= 1(8) + 1(2) + 0 = 8+2 = 10 \end{aligned}$$

$$[\vec{b},\vec{c},\vec{d}] = \begin{vmatrix} 1 & -1 & -4 \\ 0 & 3 & -1 \\ 2 & 5 & 1 \end{vmatrix} \begin{aligned} &= 1(3+5) + 1(0+2) - 4(0-6) \\ &= 1(8) + 1(2) - 4(-6) \\ &= 8+2+24 = 34 \end{aligned}$$

$[\vec{a},\vec{c},\vec{d}]\vec{b} - [\vec{b},\vec{c},\vec{d}]\vec{a}$

$= 10(\hat{i} - \hat{j} - 4\hat{k}) - (34)(\hat{i} - \hat{j}) = 10((\hat{i} - \hat{j} - 4\hat{k}) - 34(\hat{i} - \hat{j})$

$= 10\hat{i} - 10\hat{j} - 40\hat{k} - 34\hat{i} + 34\hat{j} = -24\hat{i} + 24\hat{j} - 40\hat{k}$

L.H.S = R.H.S

Exercise 6.5(4):

Show that the lines $\frac{x-3}{3} = \frac{y-3}{-1}$, z – 1 = 0.and $\frac{x-6}{2} = \frac{z-1}{3}$, y – 2 = 0 intersect. Also find the point of intersection.

Solution:

$\frac{x-3}{3} = \frac{y-3}{-1}, z-1=0 \Rightarrow \frac{x-3}{3} = \frac{y-3}{-1} = \frac{z-1}{0}$

$\frac{x-6}{2} = \frac{z-1}{3}, y-2=0 \Rightarrow \frac{x-6}{2} = \frac{y-2}{0} = \frac{z-1}{3}$

$\vec{a} = 3\hat{i} + 3\hat{j} + \hat{k}$ $\quad\vec{b} = 3\hat{i} - \hat{j} + 0\hat{k}$

$\vec{c} = 6\hat{i} + 2\hat{j} + \hat{k}$ $\quad\vec{d} = 2\hat{i} + 0\hat{j} + 3\hat{k}$

$\vec{b}$ and $\vec{d}$ are not parallel.

$\vec{c} - \vec{a} = 6\hat{i} + 2\hat{j} + \vec{k} - 3\hat{i} - 3\hat{j} - \vec{k} = 3\hat{i} - \hat{j}$

$$\vec{b} \times \vec{d} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & 0 \\ 2 & 0 & 3 \end{vmatrix} = \hat{i}(-3) - \hat{j}(9) + \hat{k}(0+2)$$

$= -3\hat{i} - 9\hat{j} + 2\hat{k}$

$(\vec{c} - \vec{a}).(\vec{b} \times \vec{d}) = (3\hat{i} - \hat{j}).(-3\hat{i} - 9\hat{j} + 2\hat{k})$

$= 3(-3)+(-1)(-9)+0$

$= 0$

The lines are intersecting

Any point on the line

$\frac{x-3}{3} = \frac{y-3}{-1} = \frac{z-1}{0} = \lambda$

$\frac{x-3}{3} = \lambda, \frac{y-3}{-1} = \lambda, \frac{z-1}{0} = \lambda$

$x - 3 = 3\lambda, y - 3 = -\lambda, z - 1 = 0$

$x = 3 + 3\lambda, y = 3 - \lambda, z = 1$

Any point $(3 + 3\lambda, 3 - \lambda, 1)$

$\frac{x-6}{2} = \frac{y-2}{0} = \frac{z-1}{3} = \mu$

$\frac{x-6}{2} = \mu, \frac{y-2}{0} = \mu, \frac{z-1}{3} = \mu$

$x - 6 = 2\mu, y - 2 = 0, z - 1 = 3\mu$

$x - 6 = 2\mu, y - 2 = 0, z - 1 = 3\mu$

$x = 2\mu + 6$, y

$= 2, z = 3\mu + 1$

any point $(2\mu + 6, 2, 3\mu + 1)$

Since line intersects for some λ and μ

$(3 + 3\lambda, 3 - \lambda, 1) = (2\mu + 6, 2, 3\mu + 1)$

$3 - \lambda = 2 \Rightarrow -\lambda = 2 - 3 = -1 \Rightarrow \lambda = 1$

$3\mu + 1 = 1 \Rightarrow 3\mu = 1 - 1 = 0, \Rightarrow \mu = 0.$

Point of intersection (6, 2, 1)

Example 6.33: **Show that the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-4}{5} = \frac{y-1}{2} = z$ intersect. Also find the point of intersection.**

Solution:

$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ $\quad\frac{x-4}{5} = \frac{y-1}{2} = z = \frac{z-0}{1}$

$\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$ $\quad\vec{b} = 2\hat{i} + 3\vec{j} + 4\hat{k}$

$\vec{c} = 4\hat{i} + \hat{j} + 0\hat{k}$ $\quad\vec{d} = 5\hat{i} + 2\hat{j} + \hat{k}$

$\vec{b}$ and $\vec{d}$ are not parallel.

$\vec{c} - \vec{a} = 4\hat{i} + \hat{j} + 0\hat{k} - \hat{i} - 2\hat{j} - 3\hat{k} = 3\hat{i} - \hat{j} - 3\hat{k}$

$$\vec{b} \times \vec{d} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 5 & 2 & 1 \end{vmatrix} = \hat{i}(3 - 8) - \hat{j}(2 - 10) + \hat{k}(4 - 15)$$

$= -5\hat{i} + 8\hat{j} - 11\hat{k}$

$(\vec{c} - \vec{a}).(\vec{b} \times \vec{d}) = (3\hat{i} - \hat{j} - 3\hat{k}).(-5\hat{i} + 8\hat{j} - 11\hat{k})$

$= 3(-5) + (-1)(8) + (-3)(-11)$

$= -15 - 8 + 33 = 0$

The lines are intersecting

Any point on the line

$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} = \lambda$

$\frac{x-1}{2} = \lambda, \frac{y-2}{3} = \lambda, \frac{z-3}{4} = \lambda$

$x - 1 = 2\lambda, y - 2 = 3\lambda, z - 3 = 4\lambda$

$x = 2\lambda + 1, y = 3\lambda + 2, z = 4\lambda + 3$

Any point $(2\lambda + 1, 3\lambda + 2, 4\lambda + 3)$

Any point on the second line

$\frac{x-4}{5} = \frac{y-1}{2} = \frac{z-0}{1} = \mu$

$\frac{x-4}{5} = \mu, \frac{y-1}{2} = \mu, \frac{z-0}{1} = \mu$

$x - 4 = 5\mu, y - 1 = 2\mu, z = \mu$

$x = 5\mu + 4, y = 2\mu + 1, z = \mu$

Any point $(5\mu + 4, 2\mu + 1, \mu)$

Since line intersects for some λ and μ

$(2\lambda + 1, 3\lambda + 2, 4\lambda + 3) = (5\mu + 4, 2\mu + 1, \mu)$

Equating x–co ordinate $2\lambda+1 = 5\mu+4$

$\Rightarrow 2\lambda - 5\mu = 3$ ______ (1)

Equating z-coordinate: $4\lambda + 3 = \mu$

$\Rightarrow 4\lambda - \mu = -3$ ______(2)

(1)x2 $\Rightarrow 4\lambda - 10\mu = 6$

(2)x1 $\Rightarrow 4\lambda - \mu = -3$

\- + +

$-9\mu = 9$

$\mu = -1$

Substitute $\mu = -1$ in $4\lambda - \mu = -3$

$4\lambda - (-1) = -3$

$4\lambda + 1 = -3$

$4\lambda = -3 - 1 = -4$

$\lambda = -1$

So point of intersection is (-1, -1, -1)

Example 6.37: **Find the coordinate of the perpendicular drawn from the point (-1, 2, 3) to the straight line $\vec{r} = (\hat{i} - 4\hat{j} + 3\hat{k}) + t(2\hat{i} + 3\hat{j} + \hat{k})$, also find the shortest distance from the given point to the straight line.**

Solution: $\vec{r} = (\hat{i} - 4\hat{j} + 3\hat{k}) + t(2\hat{i} + 3\hat{j} + \hat{k})$

$\vec{a} = \hat{i} - 4\hat{j} + 3\hat{k}$ $\quad (x_1, y_1, z_1) = (1, -4, 3)$

$\vec{b} = 2\hat{i} + 3\hat{j} + \hat{k}$ $\quad (b_1, b_2, b_3) = (2, 3, 1)$

Cartesian equation is : $\frac{x - x_1}{b_1} = \frac{y - y_1}{b_2} = \frac{z - z_1}{b_3}$

$\frac{x-1}{2} = \frac{y+4}{3} = \frac{z-3}{1}$

To find any point : $\frac{x-1}{2} = \frac{y+4}{3} = \frac{z-3}{1} = t$

$\frac{x-1}{2} = t, \frac{y+4}{3} = t, \frac{z-3}{1} = t$

$x = 2t + 1, y = 3t - 4, z = t + 3$

Any point is $(2t + 1, 3t - 4, t + 3)$

Let foot of the perpendicular $B(2t + 1, 3t - 4, t + 3)$

Point $A(-1, 2, 3)$

Direction ratios of line joining two points A and B

D.r's = $(2t + 1 + 1, 3t - 4 - 2, t + 3 - 3) = (2t + 2, 3t - 6, t)$

D.r's 0f the given line is 2, 3,1

Since lines are perpendicular:

$2(2t + 2) + 3(3t - 6) + (1)(t) = 0$

$4t + 4 + 9t - 18 + t = 0 \Rightarrow 14t - 14 = 0 \Rightarrow 14t = 14 \Rightarrow t = 1$

Point of intersection is $(2(1)+1, 3(1) - 4, 1 + 3)$

$= (3, -1, 4)$

Shortest distance of the point A from the line

$A = (-1, 2, 3)$ and $B(3, -1, 4)$

$AB = \sqrt{(3 - (-1))^2 + (-1 - 2)^2 + (4 - 3)^2}$

$= \sqrt{(3 + 1)^2 + (-3)^2 + 1^2} = \sqrt{16 + 9 + 1} = \sqrt{26}$

Example 6.35: **Determine whether the pair of straight lines $\vec{r} = (2\hat{i} + 6\hat{j} + 3\hat{k}) + t(2\hat{i} + 3\hat{j} + 4\hat{k})$ and $\vec{r} = (2\hat{j} - 3\hat{k}) + s(\hat{i} + 2\hat{j} + 3\hat{k})$ are parallel and find the shortest distance between them**

SOLUTION;

$\vec{a} = 2\hat{i} + 6\hat{j} + 3\hat{k}$ $\quad \vec{b} = 2\hat{i} + 3\hat{j} + 4\hat{k}$

$\vec{c} = 2\hat{j} - 3\hat{k}$ $\quad \vec{d} = \hat{i} + 2\hat{j} + 3\hat{k}$

clearly $\vec{b}$ is not scalar multiple of $\vec{d}$ so the vectors are not parallel and hence the lines are not parallel.

$\vec{c} - \vec{a} = 2\hat{j} - 3\hat{k} - 2\hat{i} - 6\hat{j} - 3\hat{k} = -2\hat{i} - 4\hat{j} - 6\hat{k}$

$\vec{b} \times \vec{d} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 1 & 2 & 3 \end{vmatrix} = \hat{i}(9 - 8) - \hat{j}(6 - 4) + \hat{k}(4 - 3)$

$= \hat{i} - 2\hat{j} + \hat{k}$

$(\vec{c} - \vec{a}).(\vec{b} \times \vec{d}) = (-2\hat{i} - 4\hat{j} - 6\hat{k}).(\hat{i} - 2\hat{j} + \hat{k})$

$= (-2)(1) + (-4)(-2) + (-6)(1)$

$= -2 + 8 - 6 = 0$

The lines are coplanar so lines are intersecting

so distance = 0

Example 6.34:

Find the parametric form of vector equation of a straight line passing through the point of intersection of the straight lines $\vec{r} = (\hat{i} + 3\hat{j} - \hat{k}) + t(2\hat{i} + 3\hat{j} + 2\hat{k})$ And $\frac{x-2}{1} = \frac{y-4}{2} = \frac{z+3}{4}$ and perpendicular to both straight lines.

Solution:

$\vec{r} = (\hat{i} + 3\hat{j} - \hat{k}) + t(2\hat{i} + 3\hat{j} + 2\hat{k})$

$\vec{a} = \hat{i} + 3\hat{j} - \hat{k}$ $\quad (x_1, y_1, z_1) = (1, 3, -1)$

$\vec{b} = 2\hat{i} + 3\hat{j} + 2\hat{k}$ $\quad (b_1, b_2, b_3) = (2, 3, 2)$

Cartesian equation is : $\frac{x - x_1}{b_1} = \frac{y - y_1}{b_2} = \frac{z - z_1}{b_3}$

$\frac{x-1}{2} = \frac{y-3}{3} = \frac{z+1}{2}$

To find any point : $\frac{x-1}{2} = \frac{y-3}{3} = \frac{z+1}{2} = t$

$\frac{x-1}{2} = t, \frac{y-3}{3} = t, \frac{z+1}{2} = t$

$x = 2t + 1, y = 3t + 3, z = 2t - 1$

Any point is $(2t + 1, 3t + 3, 2t - 1)$

SECOND LINE: $\frac{x-2}{2} = \frac{y-4}{2} = \frac{z+3}{4}$

To find any point let $\frac{x-2}{1} = \frac{y-4}{2} = \frac{z+3}{4} = s$

$\frac{x-2}{1} = s, \frac{y-4}{2} = s, \frac{z+3}{4} = s$

$x = s + 2, y = 2s + 4, z = 4s - 3$

Any point $(s + 2, 2s + 4, 4s - 3)$

Since lines are intersecting

$(2t + 1, 3t + 3, 2t - 1) = (s + 2, 2s + 4, 4s - 3)$

x coordinate : $2t + 1 = s + 2 \Rightarrow 2t - s = 2 - 1 \Rightarrow 2t - s = 1$

y coordinate: $3t + 3 = 2s + 4 \Rightarrow 3t - 2s = 4 - 3 \Rightarrow 3t - 2s = 1$

z coordinate : $2t - 1 = 4s - 3 \Rightarrow 2t - 4s = -3 + 1 \Rightarrow$

$2t - 4s = -2$

$2t - 4s = -2$ divide by 2 $\Rightarrow t - 2s = -1$

solving we get t = 1 and s = 1

point is $(1+2, 2(1)+4, 4(1)-3) = (3, 6, 1)$

$\vec{b} \times \vec{d} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 2 \\ 1 & 2 & 4 \end{vmatrix} = \hat{i}(12 - 4) - \hat{j}(8 - 2) + \hat{k}(4 - 3)$

$= 8\hat{i} - 6\hat{j} + \hat{k}$

Equation of line: through (3,6,1) and parallel to $8\hat{i} - 6\hat{j} + \hat{k}$

$\vec{a} = 3\hat{i} + 6\hat{j} + \hat{k}$ $\quad \vec{b} = 8\hat{i} - 6\hat{j} + \hat{k}$

Vector equation: $\vec{r} = \vec{a} + t\vec{b}, \quad t \in \mathbb{R}$

$\vec{a} = (3\hat{i} + 6\hat{j} + \hat{k}) + t(8\hat{i} - 6\hat{j} + \hat{k})$

Example 6.43

Find the non-parametric form of vector equation , and Cartesian equation of the plane passing through the point (0,1,-5) and parallel to the straight lines $\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + s(2\hat{i} + 3\hat{j} + 6\hat{k})$ and $\vec{r} = (\hat{i} - 3\hat{j} + 5\hat{k}) + t(\hat{i} + \hat{j} - \hat{k})$.

Solution:

Point : $\vec{a} = 0\hat{i} + 1\hat{j} - 5\hat{k}$ $(x_1, y_1, z_1) = (0,1,-5)$

Parallel vector : $\vec{b} = 2\hat{i} + 3\hat{j} + 6\hat{k}$ $(b_1, b_2, b_3) = (2, 3, 6)$

Parallel vector: $\vec{c} = \hat{i} + \hat{j} - \hat{k}$ $(c_1, c_2, c_3) = (1,1, -1)$

Cartesian Equation : $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$

$$\begin{vmatrix} x - 0 & y - 1 & z + 5 \\ 2 & 3 & 6 \\ 1 & 1 & -1 \end{vmatrix} = 0$$

$\Rightarrow(x)(-3-6) - (y-1)(-2-6)+(z+5)(2-3) = 0$

$\Rightarrow x(-9) - (y-1)(-8)+(z+5)(-1) = 0$

$\Rightarrow -9x +8(y-1)-1(z+5) = 0$ $\Rightarrow -9x + 8y - 8 - z -5 = 0$

$\Rightarrow$ $-9x + 8y - z = 13$

$\Rightarrow 9x - 8y + z = -13$

Non parametric vector equation:

$(x\hat{i} + y\hat{j} + z\hat{k}).(-9\hat{i} + 8\hat{j} - \hat{k}) = 13 \Rightarrow \vec{r}.(-9\hat{i} + 8\hat{j} - \hat{k}) = 13$

Example 6.44:

Find the vector parametric , vector non-parametric and Cartesian form of the equation of the plane passing through the points (-1,2,0) , (2,2,-1) and parallel to the straight line $\frac{x-1}{1} = \frac{2y+1}{2} = \frac{z+1}{-1}$.

Solution: $\frac{x-1}{1} = \frac{2y+1}{2} = \frac{z+1}{-1}$

rewritten as $\frac{x-1}{1} = \frac{2(y+\frac{1}{2})}{2} = \frac{z+1}{-1} \Rightarrow \frac{x-1}{1} = \frac{(y+\frac{1}{2})}{1} = \frac{z+1}{-1}$

Point : $\vec{a} = -1\hat{i} + 2\hat{j} + 0\hat{k}$ $(x_1, y_1, z_1) = (-1,2,0)$

Point : $\vec{b} = 2\hat{i} + 2\hat{j} - 1\hat{k}$ $(x_2, y_2, z_2) = (2, 2, -1)$

Parallel Vector : $\vec{c} = \hat{i} + \hat{j} - \hat{k}$ $(c_1, c_2, c_3) = (1, 1, -1)$

$\vec{b} - \vec{a} = 2\hat{i} + 2\hat{j} - 1\hat{k} - (-1\hat{i} + 2\hat{j} + 0\hat{k}) = 2\hat{i} + 2\hat{j} - 1\hat{k} + 1\hat{i} - 2\hat{j}$

$= 3\hat{i} + 0\hat{j} - 1\hat{k}$

Parametric Vector equation: $\vec{r} = \vec{a} + s(\vec{b} - \vec{a}) + t\vec{c}$

$\vec{r} = (-1\hat{i} + 2\hat{j} + 0\hat{k}) + s(3\hat{i} + 0\hat{j} - 1\hat{k}) + t(\hat{i} + \hat{j} - \hat{k})$

Cartesian Equation: $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$

$$\begin{vmatrix} x - (-1) & y - 2 & z - 0 \\ 2 - (-1) & 2 - 2 & -1 - 0 \\ 1 & 1 & -1 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} x + 1 & y - 2 & z \\ 3 & 0 & -1 \\ 1 & 1 & -1 \end{vmatrix} = 0$$

$\Rightarrow(x+1)(0+1)-(y-2)(-3+1)+(z)(3) = 0$

$\Rightarrow 1(x+1)+2(y-2)+3(z) = 0$ $\Rightarrow x +1+ 2y - 4+ 3z = 0$

$x + 2y + 3z - 3 = 0 \Rightarrow x + 2y + 3z = 3$

Non parametric vector equation:

$(x\hat{i} + y\hat{j} + z\hat{k}).(\hat{i} + 2\hat{j} + 3\hat{k}) = 3 \Rightarrow \vec{r}.(\hat{i} + 2\hat{j} + 3\hat{k}) = 3$

Exercise 6.7(1)

Find the non-parametric form of vector equation and Cartesian equation of the plane passing through the point (2,3,6) and parallel to the straight lines $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-3}{1}$ and $\frac{x+3}{2} = \frac{y-3}{-5} = \frac{z+1}{-3}$.

Solution:

Point : $\vec{a} = 2\hat{i} + 3\hat{j} + 6\hat{k}$ $(x_1, y_1, z_1) = (2,3,6)$

Parallel vector : $\vec{b} = 2\hat{i} + 3\hat{j} + 1\hat{k}$ $(b_1, b_2, b_3) = (2, 3, 1)$

Parallel vector:: $\vec{c} = 2\hat{i} - 5\hat{j} - 3\hat{k}$ $(c_1, c_2, c_3) = (2, -5, -3)$

Cartesian equation : $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$

$$\begin{vmatrix} x - 2 & y - 3 & z - 6 \\ 2 & 3 & 1 \\ 2 & -5 & -3 \end{vmatrix} = 0$$

$\Rightarrow(x-2)(-9+5) - (y-3)(-6-2)+(z-6)(-10-6) = 0$

$\Rightarrow(x-2)(-4) - (y - 3)(-8)+(z - 6)(-16) = 0$

$\Rightarrow -4(x-2) +8(y-3)-16(z - 6) = 0$

$\Rightarrow -4x + 8 + 8y - 24 - 16z + 96 = 0$

$\Rightarrow x - 2y + 4z + 80 = 0$ $\Rightarrow x - 2y + 4z = 20$

Non parametric vector equation:

$(x\hat{i} + y\hat{j} + z\hat{k}).(\hat{i} - 2\hat{j} + 4\hat{k}) = 20$ $\Rightarrow \vec{r}.(\hat{i} - 2\hat{j} + 4\hat{k}) = 20$

Exercise 6.7 (2):

Find the parametric form of vector equation , and Cartesian equations of the plane passing through the points (2,2,1) , (9,3,6) and perpendicular to the plane $2x + 6y + 6z = 9$.

Solution:

Point : $\vec{a} = 2\hat{i} + 2\hat{j} + \hat{k}$ $(x_1, y_1, z_1) = (2,2, 1)$

Point : $\vec{b} = 9\hat{i} + 3\hat{j} + 6\hat{k}$ $(x_2, y_2, z_2) = (9, 3, 6)$

Parallel Vector : $\vec{c} = 2\hat{i} + 6\hat{j} + 6\hat{k}$ $(c_1, c_2, c_3) = (2, 6, 6)$

$\vec{b} - \vec{a} = 9\hat{i} + 3\hat{j} + 6\hat{k} - (2\hat{i} + 2\hat{j} + \hat{k})$

$= 9\hat{i} + 3\hat{j} + 6\hat{k} - 2\hat{i} - 2\hat{j} - \hat{k} = 7\hat{i} + 1\hat{j} + 5\hat{k}$

Parametric Vector equation: $\vec{r} = \vec{a} + s(\vec{b} - \vec{a}) + t\vec{c}$

$\vec{r} = (2\hat{i} + 2\hat{j} + \hat{k}) + s(7\hat{i} + 1\hat{j} + 5\hat{k}) + t(2\hat{i} + 6\hat{j} + 6\hat{k})$

Cartesian Equation:: $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$

$$\begin{vmatrix} x - 2 & y - 2 & z - 1 \\ 9 - 2 & 3 - 2 & 6 - 1 \\ 2 & 6 & 6 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} x - 2 & y - 2 & z - 1 \\ 7 & 1 & 5 \\ 2 & 6 & 6 \end{vmatrix} = 0$$

$\Rightarrow(x- 2)(6 - 30) - (y - 2)(42 - 10) + (z - 1)(42 - 2) = 0$

$\Rightarrow -24(x - 2) - 32(y-2) + 40(z - 1) = 0$

$\Rightarrow -24x + 48 -32y + 64 +40z - 40 = 0$

$\Rightarrow -24x - 32y + 40z + 72 = 0$

$\Rightarrow 3x + 4y - 5z - 9 = 0$

Non parametric vector equation:

$(x\hat{i} + y\hat{j} + z\hat{k}).(3\hat{i} + 4\hat{j} - 5\hat{k}) - 9 = 0$

$\Rightarrow \vec{r}.(3\hat{i} + 4\hat{j} - 5\hat{k}) = 9$

Exercise 6.7(4)

Find the non-parametric form of vector equation and Cartesian equation of the plane passing through the point (1,-2,4) and perpendicular to the plane $x + 2y - 3z = 11$ and parallel to the line $\frac{x+7}{3} = \frac{y+3}{-1} = \frac{z}{1}$

Solution: $(1,-2,4)$ $\quad x + 2y - 3z = 11 \quad \frac{x+7}{3} = \frac{y+3}{-1} = \frac{z}{1}$

Point : $\vec{a} = \hat{i} - 2\hat{j} + 4\hat{k}$ $(x_1, y_1, z_1) = (1, -2, 4)$

Parallel Vector: $\vec{b} = \hat{i} + 2\hat{j} - 3\hat{k}$ $(b_1, b_2, b_3) = (1, 2, -3)$

Parallel Vector: $\vec{c} = 3\hat{i} - \hat{j} + \hat{k}$ $(c_1, c_2, c_3) = (3, -1, 1)$

Parametric Vector equation: $\vec{r} = \vec{a} + s\vec{b} + t\vec{c}$

$\Rightarrow \vec{r} = (\hat{i} - 2\hat{j} + 4\hat{k}) + s(\hat{i} + 2\hat{j} - 3\hat{k}) + t(3\hat{i} - \hat{j} + \hat{k})$

Cartesian Equation: $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$

$$\begin{vmatrix} x-1 & y+2 & z-4 \\ 1 & 2 & -3 \\ 3 & -1 & 1 \end{vmatrix} = 0$$

$\Rightarrow (x - 1)(2-3) - (y+2)(1+9) + (z-4)(-1-6) = 0$

$\Rightarrow (x-1)(-1) - (y+2)(10) + (z-4)(-7) = 0$

$\Rightarrow -1(x -1) -10(y + 2) -7(z - 4) = 0$

$\Rightarrow -x + 1 - 10y - 20 - 7z + 28 = 0 \quad \Rightarrow -x - 10y - 7z + 9 = 0$

$\Rightarrow x + 10y + 7z - 9 = 0$

Non Parametric Vector Equation:

$(x\hat{i} + y\hat{j} + z\hat{k}).(\hat{i} + 10\hat{j} + 7\hat{k}) - 9 = 0$

$\Rightarrow \vec{r}.(\hat{i} + 10\hat{j} + 7\hat{k}) - 9 = 0$

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3. Find parametric form of vector equation and Cartesian equations of the plane passing through the points (2,2,1), (1,-2,3) and parallel to the straight line passing through the points (2,1,-3) and (-1,5,-8).

Solution: $\overrightarrow{OP} = 2\hat{i} + \hat{j} - 3\hat{k}$ $\quad \overrightarrow{OQ} = -\hat{i} + 5\hat{j} - 8\hat{k}$

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = -\hat{i} + 5\hat{j} - 8\hat{k} - 2\hat{i} - \hat{j} + 3\hat{k} = -3\hat{i} + 4\hat{j} - 5\hat{k}$

Point : $\vec{a} = 2\hat{i} + 2\hat{j} + \hat{k}$ $\quad (x_1, y_1, z_1) = (2, 2, 1)$

Point : $\vec{b} = \hat{i} - 2\hat{j} + 3\hat{k}$ $\quad (x_2, y_2, z_2) = (1, -2, 3)$

Parallel Vector : $\vec{c} = -3\hat{i} + 4\hat{j} - 5\hat{k}$ $\quad (c_1, c_2, c_3) = (-3, 4, -5)$

$\vec{b} - \vec{a} = \hat{i} - 2\hat{j} + 3\hat{k} - (2\hat{i} + 2\hat{j} + \hat{k})$

$= \hat{i} - 2\hat{j} + 3\hat{k} - 2\hat{i} - 2\hat{j} - 3\hat{k} = -\hat{i} - 4\hat{j} + 2\hat{k}$

Parametric Vector Equation: $\vec{r} = \vec{a} + s(\vec{b} - \vec{a}) + t\vec{c}$

$\vec{r} = (2\hat{i} + 2\hat{j} + \hat{k}) + s(-\hat{i} - 4\hat{j} + 2\hat{k}) + t(-3\hat{i} + 4\hat{j} - 5\hat{k})$

Cartesian Equation : $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$

$$\begin{vmatrix} x-2 & y-2 & z-1 \\ 1-2 & -2-2 & 3-1 \\ -3 & 4 & -5 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-2 & y-2 & z-1 \\ -1 & -4 & 2 \\ -3 & 4 & -5 \end{vmatrix} = 0$$

$\Rightarrow (x - 2)(20 - 8) - (y-2)(5 -+ 6) + (z - 1)(-4 - 12) = 0$

$\Rightarrow 12(x-2) - 11(y-2) - 16(z - 1) = 0$

$\Rightarrow 12x - 24 - 11y + 22 - 16z + 16 = 0$

$\Rightarrow 12x - 11y - 16z + 14 = 0 \Rightarrow 12x - 11y - 16z = -14$

Non Parametric Vector Equation:

$(x\hat{i} + y\hat{j} + z\hat{k}).(12\hat{i} - 11\hat{j} - 16\hat{k}) = -14$

$\Rightarrow \vec{r}.(12\hat{i} - 11\hat{j} - 16\hat{k}) = -14$

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5. Find the parametric form of vector equation, and Cartesian equations of the plane containing the line $\vec{r} = (\hat{i} - \hat{j} + 3\hat{k}) + t(2\hat{i} - \hat{j} + 4\hat{k})$ and perpendicular to plane $\vec{r} \cdot (\hat{i} + 2\hat{j} + \hat{k}) = 8$.

Solution:

containing the line $\vec{r} = (\hat{i} - \hat{j} + 3\hat{k}) + t(2\hat{i} - \hat{j} + 4\hat{k})$

Point : $\vec{a} = \hat{i} - \hat{j} + 3\hat{k}$ $\quad (x_1, y_1, z_1) = (1, -1, 3)$

Parallel Vector: $\vec{b} = 2\hat{i} - \hat{j} + 4\hat{k}$ $\quad (b_1, b_2, b_3) = (2, -1, 4)$

Parallel Vector: $\vec{c} = \hat{i} + 2\hat{j} + \hat{k}$ $\quad (c_1, c_2, c_3) = (1, 2, 1)$

Parametric Vector equation: $\vec{r} = \vec{a} + s\vec{b} + t\vec{c}$

$\Rightarrow \vec{r} = (\hat{i} - \hat{j} + 3\hat{k}) + s(2\hat{i} - \hat{j} + 4\hat{k}) + t(\hat{i} + 2\hat{j} + \hat{k})$

Cartesian Equation: $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$

$$\begin{vmatrix} x-1 & y+1 & z-3 \\ 2 & -1 & 4 \\ 1 & 2 & 1 \end{vmatrix} = 0$$

$\Rightarrow (x - 1)(-1-8) - (y + 1)(2 - 4) + (z - 3)(4 + 1) = 0$

$\Rightarrow (x-1)(-9) - (y + 1)(-2) + (z - 3)(5) = 0$

$\Rightarrow -9(x-1) + 2(y + 1) + 5(z - 3) = 0$

$\Rightarrow -9x + 9 + 2y + 2 + 5z - 15 = 0$

$\Rightarrow -9x + 2y + 5z - 4 = 0 \Rightarrow 9x - 2y - 5z + 4 = 0$

Non Parametric Vector Equation:

$(x\hat{i} + y\hat{j} + z\hat{k}).(9\hat{i} - 2\hat{j} - 5\hat{k}) + 4 = 0$

$\Rightarrow \vec{r}.(9\hat{i} - 2\hat{j} - 5\hat{k}) + 4 = 0$

---

6. Find the parametric vector, non-parametric vector and Cartesian form of the equations of the plane passing through the three non-collinear points (3,6,-2), (-1,-2,6), (6, 4,-2).

Solution:

Point : $\vec{a} = 3\hat{i} + 6\hat{j} - 2\hat{k}$ $\quad (x_1, y_1, z_1) = (3, 6, -2)$

Point : $\vec{b} = -\hat{i} - 2\hat{j} + 6\hat{k}$, $(x_2, y_2, z_2) = (-1, -2, 6)$

Point : $\vec{c} = 6\hat{i} + 4\hat{j} - 2\hat{k}$, $(x_3, y_3, z_3) = (6, 4, -2)$

$\vec{b} - \vec{a} = -\hat{i} - 2\hat{j} + 6\hat{k} - 3\hat{i} - 6\hat{j} + 2\hat{k} = -4\hat{i} - 8\hat{j} + 8\hat{k}$

$\vec{c} - \vec{a} = 6\hat{i} + 4\hat{j} - 2\hat{k} - 3\hat{i} - 6\hat{j} + 2\hat{k} = 3\hat{i} - 2\hat{j} + 0\hat{k}$

Parametric Vector Equation:

$\vec{r} = \vec{a} + s(\vec{b} - \vec{a}) + t(\vec{c} - \vec{a})$

$\vec{r} = (3\hat{i} + 6\hat{j} - 2\hat{k}) + s(-4\hat{i} - 8\hat{j} + 8\hat{k}) + t(3\hat{i} - 2\hat{j} + 0\hat{k})$

Cartesian Equation : $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$

$$\begin{vmatrix} x-3 & y-6 & z+2 \\ -1-3 & -2-6 & 6+2 \\ 6-3 & 4-6 & -2+2 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-3 & y-6 & z+2 \\ -4 & -8 & 8 \\ 3 & -2 & 0 \end{vmatrix} = 0$$

$\Rightarrow (x - 3)(0 + 16) - (y - 6)(0 - 24) + (z + 2)(8 + 24) = 0$

$\Rightarrow 16(x-3) + 24(y-6) + 32(z + 2) = 0$

$\Rightarrow 16x - 48 + 24y - 144 + 32z + 64 = 0$

$\Rightarrow 16x + 24y + 32z - 128 = 0 \Rightarrow 2x + 3y + 4z - 16 = 0$

Non Parametric Vector Equation:

$(x\hat{i} + y\hat{j} + z\hat{k}).(2\hat{i} + 3\hat{j} + 4\hat{k}) = 16$

$\Rightarrow \vec{r}.(2\hat{i} + 3\hat{j} + 4\hat{k}) = 16$

Example 6.46

Show that the lines $\vec{r} = (-\hat{i} - 3\hat{j} - 5\hat{k}) + s(3\hat{i} + 5\hat{j} + 7\hat{k})$ and $\vec{r} = (2\hat{i} + 4\hat{j} + 6\hat{k}) + t(\hat{i} + 4\hat{j} + 7\hat{k})$ are coplanar. Also,find the non-parametric form of vector equation of the plane containing these lines.

Solution

$\vec{r} = \vec{a} + t\vec{b}$, $\qquad \vec{r} = \vec{c} + s\vec{d}$

$\vec{a} = -\hat{i} - 3\hat{j} - 5\hat{k}$, $\qquad \vec{b} = 3\hat{i} + 5\hat{j} + 7\hat{k}$,

$\vec{c} = 2\hat{i} + 4\hat{j} + 6\hat{k}$ and $\qquad \vec{d} = \hat{i} + 4\hat{j} + 7\hat{k}$

We know that the two given lines are coplanar, if

$(\vec{c} - \vec{a}) \cdot (\vec{b} \times \vec{d}) = 0$

$$\vec{b} \times \vec{d} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 5 & 7 \\ 1 & 4 & 7 \end{vmatrix} = 7\hat{i} - 14\hat{j} + 7\hat{k}$$

$\vec{c} - \vec{a} = 2\hat{i} + 4\hat{j} + 6\hat{k} - (-\hat{i} - 3\hat{j} - 5\hat{k})$

$\qquad = 2\hat{i} + 4\hat{j} + 6\hat{k} + \hat{i} + 3\hat{j} + 5\hat{k} = 3\hat{i} + 7\hat{j} + 11\hat{k}$

$(\vec{c} - \vec{a}) \cdot (\vec{b} \times \vec{d}) = (3\hat{i} + 7\hat{j} + 11\hat{k}) \cdot (7\hat{i} - 14\hat{j} + 7\hat{k})$

$= 21 - 98 + 77 = 98 - 98 = 0$

$(\vec{r} - \vec{a}) \cdot (\vec{b} \times \vec{d}) = 0$

$(\vec{r} - (-\hat{i} - 3\hat{j} - 5\hat{k})) \cdot (7\hat{i} - 14\hat{j} + 7\hat{k}) = 0.$

$\vec{r} \cdot (7\hat{i} - 14\hat{j} + 7\hat{k}) - (-\hat{i} - 3\hat{j} - 5\hat{k})) \cdot (7\hat{i} - 14\hat{j} + 7\hat{k}) = 0$

$\vec{r} \cdot (7\hat{i} - 14\hat{j} + 7\hat{k}) - (-7 + 42 - 35) = 0$

$\vec{r} \cdot (7\hat{i} - 14\hat{j} + 7\hat{k}) = 0 \quad (\div \text{ by } 7) \quad \vec{r} \cdot (\hat{i} - 2\hat{j} + \hat{k}) = 0$

EX 6.8 (1).

Show that the straight lines $\vec{r} = (5\hat{i} + 7\hat{j} - 3\hat{k}) + s(4\hat{i} + 4\hat{j} - 5\hat{k})$ and $\vec{r} = (8\hat{i} + 4\hat{j} + 5\hat{k}) + t(7\hat{i} + \hat{j} + 3\hat{k})$ are coplanar. Find the vector equation of the plane in which they lie.

Solution:

Let $\vec{a} = 5\vec{i} + 7\vec{j} - 3\vec{k}$ $\qquad \vec{b} = 4\vec{i} + 4\vec{j} - 5\vec{k}$

$\vec{c} = 8\vec{i} + 4\vec{j} + 5\vec{k}$ $\qquad \vec{d} = 7\vec{i} + \vec{j} + 3\vec{k}$

For coplanar $(\vec{c} - \vec{a}) \cdot (\vec{b} \times \vec{d}) = 0$

$$\vec{b} \times \vec{d} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 4 & -5 \\ 7 & 1 & 3 \end{vmatrix} = \vec{i}(12 + 5) - \vec{j}(12 + 35) + \vec{k}(4 - 28)$$

$\vec{b} \times \vec{d} = 17\vec{i} - 47\vec{j} - 24\vec{k}$

$\vec{c} - \vec{a} = (8\vec{i} + 4\vec{j} + 5\vec{k}) - (5\vec{i} + 7\vec{j} - 3\vec{k}) = 3\vec{i} - 3\vec{j} + 8\vec{k}$

$(1) \Rightarrow (3\vec{i} - 3\vec{j} + 8\vec{k}) \cdot (17\vec{i} - 47\vec{j} - 24\vec{k}) = 51 + 141 - 192 = 0$

$\therefore$ The two given lines are coplanar so,

the non-parametric vector equation is $(\vec{r} - \vec{a}) \cdot (\vec{b} \times \vec{d}) = 0$

$\vec{r} \cdot (\vec{b} \times \vec{d}) = \vec{a} \cdot (\vec{b} \times \vec{d})$

$\vec{r} \cdot (17\vec{i} - 47\vec{j} - 24\vec{k}) = (5\vec{i} + 7\vec{j} - 3\vec{k}).(17\vec{i} - 47\vec{j} - 24\vec{k})$

$\vec{r} \cdot (17\vec{i} - 47\vec{j} - 24\vec{k}) = 85 - 329 + 72$

$\Rightarrow \vec{r} \cdot (17\vec{i} - 47\vec{j} - 24\vec{k}) = -172$

EX 6.8 (2).

Show that lines $\frac{x-2}{1} = \frac{y-3}{1} = \frac{z-4}{3}$ and $\frac{x-1}{-3} = \frac{y-4}{2} = \frac{z-5}{1}$ are coplanar. Also, find the plane containing these lines.

Solution: From the lines we have,

$(x_1, y_1, z_1) = (2,3,4)$ & $(x_2, y_2, z_2) = (1,4,5)$

$(b_1, b_2, b_3) = (1,1,3)$ &$(d_1, d_2, d_3) = (-3,2,1)$

Condition for coplanarity: $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$

$$= \begin{vmatrix} -1 & 1 & 1 \\ 1 & 1 & 3 \\ -3 & 2 & 1 \end{vmatrix} = -(1 - 6) - 1(1 + 9) + 1(2 + 3)$$

$= 5 - 10 + 5 = 0$

$\therefore$ The given two lines are coplanar.

Cartesian form of equation of the plane containing the two given coplanar lines.

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} x - 2 & y - 3 & z - 4 \\ 1 & 1 & 3 \\ -3 & 2 & 1 \end{vmatrix} = 0$$

$(x - 2)[1 - 6] - (y - 3)[1 + 9] + (z - 4)[2 + 3] = 0$

$-5(x - 2) - 10(y - 3) + 5(z - 4) = 0$

$-5x + 10 - 10y + 30 + 5z - 20 = 0$

$-5x - 10y + 5z + 20 = 0$

$(\div \text{ by } -5) \Rightarrow x + 2y - z - 4 = 0$

EX 6.8 (4).

If the straight lines $\frac{x-1}{2} = \frac{y+1}{\lambda} = \frac{z}{2}$ and $\frac{x+1}{5} = \frac{y+1}{2} = \frac{z}{\lambda}$: are coplanar, find λ and equation of the planes containing these two lines.

Solution: From the lines we have,

$(x_1, y_1, z_1) = (1, -1, 0)$ and $(x_2, y_2, z_2) = (-1, -1, 0)$

$(b_1, b_2, b_3) = (2, \lambda, 2)$ and $(d_1, d_2, d_3) = (5, 2, \lambda)$

Condition for coplanarity

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$$

$$\begin{vmatrix} -2 & 0 & 0 \\ 2 & \lambda & 2 \\ 5 & 2 & \lambda \end{vmatrix} = 0 \Rightarrow -2(\lambda^2 - 4) = 0, \lambda^2 = 4 \Rightarrow \lambda = \pm 2$$

(i) If $\lambda = 2$ $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$

$$\begin{vmatrix} x - 1 & y + 1 & z \\ 2 & 2 & 2 \\ 5 & 2 & 2 \end{vmatrix} = 0$$

$(x - 1)[0] - (y + 1)[4 - 10] + z[4 - 10] = 0$

$6(y + 1) - 6(z) = 0 \Rightarrow 6y + 6 - 6z = 0$

$(\div \text{ by } 6) \Rightarrow (y - z + 1) = 0$

(ii) If $\lambda = -2$ $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$

$$\begin{vmatrix} x - 1 & y + 1 & z \\ 2 & -2 & 2 \\ 5 & 2 & -2 \end{vmatrix} = 0$$

$(x - 1)[0] - (y + 1)[-4 - 10] + z[4 + 10] = 0$

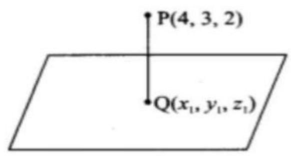
$14(y + 1) + 14z = 0 \Rightarrow 14y + 14 + 14z = 0$

$(\div \text{ by } 14) \Rightarrow y + z + 1 = 0$

EX 6.9 (8).

Find the coordinates of the foot of the perpendicular and length of the perpendicular from the point $(4, 3, 2)$ to the plane $x + 2y + 3z = 2$

Solution:



Direction of the normal plane (1,2,3)

d.c.s of the PQ is (1,2,3) $\therefore$Eqn of PQ; $\frac{x_1 - 4}{1} = \frac{y_1 - 3}{2} = \frac{z_1 - 2}{3} = k$

$x_1 = k + 4, y_1 = 2k + 3, z_1 = 3k + 2$

This passes through the plane $x + 2y + 3z = 2$

$k + 4 + 2(2k + 3) + 3(3k + 2) = 2$

$k + 4 + 4k + 6 + 9k + 6 = 2$

$14k = 2 - 16 \Rightarrow 14k = -14 \Rightarrow k = -1$

$\therefore$ The coordinate of the foot of the perpendicular is $(3, 1, -1)$

$\therefore$ Length of the perpendicular to the plane is

$$= \left| \frac{4 + 2(3) + 3(2) - 2}{\sqrt{(1)^2 + (2)^2 + (3)^2}} \right| = \left| \frac{4 + 6 + 6 - 2}{\sqrt{1 + 4 + 9}} \right| = \frac{14}{\sqrt{14}} = \sqrt{14} \text{ units}$$

CHAPTER .11 PROBABILITY DISTRIBUTION
(2 MARKS, 3 MARKS)

2 - MARKS

EXERCISE 11.1(1)

Suppose X is the number of tails occurred when three fair coins are tossed once simultaneously. Find the values of the random variable X and number of points in its inverse images.

Solution: Number of coins = 3 $n(s) = 2^3 = 8$

S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

Let X be the discrete random variable denoting no of tails

X = { 0, 1, 2 , 3}

Values of X	0	1	2	3	total
No. of elements in inverse images	1	3	3	1	8

EXERCISE 11.2 -

1. Three fair coins are tossed simultaneously. Find the probability mass function for number of heads occurred.

SOLUTION:

When three coins are tossed, the sample space is

S = { HHH, HHT, HTH, HTT, THH, THT, TTH, TTT }

' X ' is the random variable denotes the number of heads.

∴'X ' can taKe the values of 0, 1, 2 and 3

$P(X = 0) = P(\text{No heads}) = \frac{1}{8}$

$P(X = 1) = P(1 \text{ head}) = \frac{3}{8}$

$P(X = 2) = P(2 \text{ heads}) = \frac{3}{8}$

$P(X = 3) = P(3 \text{ heads}) = \frac{1}{8}$

The probability mass function is

$$f(x) = \begin{cases} 1/8 & \text{for } x = 0.3 \\ 3/8 & \text{for } x = 1, 2 \end{cases}$$

EXERCISE 11.3

1. The probability density function of X is given by $f(x) = \begin{cases} kxe^{-2x} & \text{for } x > 0 \\ 0 & \text{for } x < 0 \end{cases}$ Find the value of k .

SOLUTION: Since f(X) is a pdf $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$\int_0^{\infty} k\,x\,e^{-2x}\,dx = 1$ (since x > 0)

$k\frac{1}{2^2} = 1 \Rightarrow k\,\frac{1}{4} = 1 \Rightarrow k = 4$

EXERCISE 11.4

5. A commuter train arrives punctually at a station every half hour. Each morning, a student leaves his house to the train station. Let X denote the amount of time, in minutes, that the student waits for the train from the time he reaches the train station. It is known that the pdf of X is $f(x) = \begin{cases} \frac{1}{30} & 0 < x < 30 \\ 0 & \text{elsewhere} \end{cases}$ Obtain and interpret the expected value of the random variable X .

SOLUTION: $f(x) = \begin{cases} \frac{1}{30} & 0 < x < 30 \\ 0 & \text{elsewhere} \end{cases}$

$E(x) = \int_{-\infty}^{\infty} xf(x)\,dx$

$E(X) = \int_0^{30} x\frac{1}{30}\,dx = \frac{1}{30}\left[\frac{x^2}{2}\right]_0^{30} = \frac{1}{30}\left[\frac{30^2}{2} - 0\right] = \frac{1}{30}\left[\frac{900}{2}\right]$

$= 15$

6. The time to failure in thousands of hours of an electronic equipment used in a manufactured computer has the density function $f(x) = \begin{cases} 3e^{-3x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$ Find the expected life of this electronic equipment.

SOLUTION: $f(x) = \begin{cases} 3e^{-3x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$

$E(x) = \int_{-\infty}^{\infty} xf(x)\,dx$ $\left[\int_0^{\infty} x^n e^{-ax}dx = \frac{n!}{a^{n+1}}\right]$

$= 3\int_0^{\infty} xe^{-3x}dx = 3\frac{1!}{3^{1+1}} = 3\frac{1!}{3^2} = = \frac{1}{3}$ [n = 1, a = 3]

EXERCISE 11.5

1. Compute P(X = k) for the binomial distribution, B(n , p) where

(i) $n = 6$, $p = \frac{1}{3}$, $k = 3$

SOLUTION : $n = 6, p = \frac{1}{3}$; $k = 3$

$q = 1 - p = 1 - \frac{1}{3} = \frac{3-1}{3} = \frac{2}{3}$

$P(X = x) = n\,C_x\,p^x\,q^{n-x}$ [n = 4 , x = 3, n- x = 4-3 = 1]

$P(X = 3) = 6C_3\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^3 = 20\left(\frac{1}{27}\right)\left(\frac{8}{27}\right) = \frac{160}{729}$

(ii) $n = 10$, $p = \frac{1}{5}$, $k = 4$

$n = 10, p = \frac{1}{5}$; $k = 4$

$q = 1 - p = 1 - \frac{1}{5} = \frac{5-1}{5} = \frac{4}{5}$

$P(X = x) = n\,C_x\,p^x\,q^{n-x}$ [n = 10 , x = 4, n- x = 10-4 = 6]

$P(X = 4) = 10C_4\left(\frac{1}{5}\right)^4\left(\frac{4}{5}\right)^6$

(iii) $n = 9$, $p = \frac{1}{2}$, $k = 7$

$n = 9, p = \frac{1}{2}$; $k = 7$

$q = 1 - p = 1 - \frac{1}{2} = \frac{2-1}{2} = \frac{1}{2}$

$P(X = x) = n\,C_x\,p^x\,q^{n-x}$ [n = 9 , x = 7, n- x = 9 -7 = 2]

$P(X = 7) = 9C_7\left(\frac{1}{2}\right)^7\left(\frac{1}{2}\right)^2$

3 Using binomial distribution find the mean and variance of X for the following experiments

(i) A fair coin is tossed 100 times, and X denote the number of heads.

SOLUTION: $n = 100$ $p = \frac{1}{2}$ $q = \frac{1}{2}$

Mean = n p = $100\left(\frac{1}{2}\right) = 50$

Variance = npq = $100\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = 25$

(ii) A fair die is tossed 240 times, and X denote the number of times that four appeared.

S = { 1, 2, 3, 4, 5, 6 } $n = 240$ $p = \frac{1}{6}$ $q = 1 - \frac{1}{6} = \frac{5}{6}$

Mean = n p = $240\left(\frac{1}{6}\right) = 40$

Variance = npq = $240\left(\frac{1}{6}\right)\left(\frac{5}{6}\right) = \frac{100}{3}$

4.The probability that a certain kind of component will survive a electrical test is $\frac{3}{4}$. Find the probability that exactly 3 of the 5 components tested survive.

SOLUTION:

$n = 5$; $p = \frac{3}{4}$; $q = 1 - \frac{3}{4} = \frac{4-3}{4} = \frac{1}{4}$; $x = 3$; $5C_3 = \frac{5.4.3}{3.2.1} = 10$

$P(X = x) = n\,C_x\,p^x\,q^{n-x}$ [n = 5 , x = 3, n- x = 5 - 3 = 2]

$P(X = 3) = 5C_3\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^2 = 10\frac{3^3}{4^5} = \frac{270}{1024}$

3 - MARKS

EXERCISE 11.1: (2) **In a pack of 52 playing cards, two cards are drawn at random simultaneously. If the number of black cards drawn is a random variable, find the values of the random variable and number of points in its inverse images.**

SOLUTION: No of cards = 52; No of cards drawn = 2;

Total number of points = $52C_2 = \frac{52 \times 51}{2 \times 1} = 1326$

X be the discrete random variable denoting number of black cards

x = { 0, 1, 2}

$X(0) = X(2 \text{ Red cards}) = 26C_2 = \frac{26 \times 25}{2 \times 1} = 325$

$X(1) = X(1 \text{ Red, } 1 \text{ Black}) = 26C_1 \times 26C_1 = 26 \times 26 = 676$

$X(2) = X(2 \text{ Black cards}) = 26C_2 = \frac{26 \times 25}{2 \times 1} = 325$

Values of X	0	1	2	total
No. of elements in inverse images	325	676	325	1326

EXERCISE 11.2

2. A six sided die is marked '1' on one face, '3' on two of its faces, and '5' on remaining three faces. The die is thrown twice. If X denotes the total score in two throws, find
   (i) the probability mass function
   (ii) the cumulative distribution function
   (iii) $P(4 \le X < 10)$ (iv) $P(X \ge 6)$

Solution:

+	1	3	3	5	5	5
1	2	4	4	6	6	6
3	4	6	6	8	8	8
3	4	6	6	8	8	8
5	6	8	8	10	10	10
5	6	8	8	10	10	10
5	6	8	8	10	10	10

Given that die is marked ' 1 ' on one face, ' 3 ' on two of its faces and ' 5 ' on remaining three faces. i.e., $\{1, 3, 3, 5, 5, 5\}$ in a single die.

$P(X = 2) = \frac{1}{36}$; $P(X = 4) = \frac{4}{36}$; $P(X = 6) = \frac{10}{36}$;

$P(X = 8) = \frac{12}{36}$; $P(X = 10) = \frac{9}{36}$

(i) Probability mass function:

x	2	4	6	8	10	Total
f(x)	$\frac{1}{36}$	$\frac{4}{36}$	$\frac{10}{36}$	$\frac{12}{36}$	$\frac{9}{36}$	1

(ii) The Cumulative distribution function:

$$F(x) = \begin{cases} 0 & \text{for} \quad x < 2 \\ 1/36 & \text{for} \quad x \le 2 \\ 5/36 & \text{for} \quad x \le 4 \\ 15/36 & \text{for} \quad x \le 6 \\ 27/36 & \text{for} \quad x \le 8 \\ 1 & \text{for} \quad x \le 10 \end{cases}$$

(iii) $P(4 \le X < 10) = P(X = 4) + P(X = 6) + P(X = 8)$

$= \frac{4}{36} + \frac{10}{36} + \frac{12}{36} = \frac{26}{36} = \frac{13}{18}$

(iv) $P(X \ge 6) = P(X = 6) + P(X = 8) + P(X = 10)$

$= \frac{10}{36} + \frac{12}{36} + \frac{9}{36} = \frac{31}{36}$

Exercise 11.4

Question 1.

For the random vaniable X with the given probability mass function as below, find the mean and variance

(i) $f(x) = \begin{cases} \frac{1}{10} & x = 2, 5 \\ \frac{1}{5} & x = 0, 1, 3, 4 \end{cases}$

Solution:

(i) Given probability mass function

$f(x) = \begin{cases} \frac{1}{10} & x = 2, 5 \\ \frac{1}{5} & x = 0, 1, 3, 4 \end{cases}$

x	0	1	2	3	4	5
f(x)	1/5	1/5	1/10	1/5	1/5	1/10

Mean $E(X) = \Sigma x f(x) = 0 + \frac{1}{5} + \frac{1}{5} + \frac{3}{5} + \frac{4}{5} + \frac{1}{2}$

$= \frac{9}{5} + \frac{1}{2} = \frac{18+5}{10} = \frac{23}{10} = 2.3$

$E(X^2) = \Sigma x^2 f(x)$

$= 0 + 1^2(\frac{1}{5}) + 2^2(\frac{1}{10}) + 3^2(\frac{1}{5}) + 4^2(\frac{1}{5}) + 5^2(\frac{1}{10})$

$= 0 + \frac{1}{5} + \frac{4}{10} + \frac{9}{5} + \frac{16}{5} + \frac{25}{10} = \frac{1}{5} + \frac{2}{5} + \frac{9}{5} + \frac{16}{5} + \frac{5}{2} = \frac{28}{5} + \frac{5}{2}$

$= \frac{56+25}{10} = \frac{81}{10}$

Variance $Var(X) = E(X^2) - [E(X)]^2$

$= \frac{81}{10} - \frac{529}{100} = \frac{810-529}{100} = \frac{281}{100} = 2.81$

(ii) Given probability mass function

$f(x) = \begin{cases} \frac{4-x}{6}, x = 1, 2, 3 \end{cases}$

x	1	2	3
f(x)	1/2	1/3	1/6

Mean $E(X) = \Sigma x f(x) = \frac{1}{2} + \frac{2}{3} + \frac{1}{2} = 1 + \frac{2}{3} = \frac{5}{3} = 1.67$

$E(X^2) = \Sigma x^2 f(x) = 1^2(\frac{1}{2}) + 2^2(\frac{1}{3}) + 3^2(\frac{1}{6})$

$= \frac{1}{2} + \frac{4}{3} + \frac{9}{6} = \frac{3+8+9}{6} = \frac{20}{6} = \frac{10}{3}$

Variance $Var(X) = E(X^2) - [E(X)]^2$

$= \frac{10}{3} - \left(\frac{5}{3}\right)^2 = \frac{10}{3} - \frac{25}{9} = \frac{30-25}{9} = \frac{5}{9} = 0.56$

(iii) Given probability mass function

$f(x) = \begin{cases} 2(x-1), & 1 < x < 2 \\ 0, & \text{otherwise} \end{cases}$

Here ' X ' is a continuous random variable

Mean $E(X) = \int_{-\infty}^{\infty} x f(x) dx$

$= 2\int_1^2 x(x-1)dx = 2\int_1^2 (x^2 - x)dx = 2\left[\frac{x^3}{3} - \frac{x^2}{2}\right]_1^2$

$= 2\left[\left(\frac{8}{3} - 2\right) - \left(\frac{1}{3} - \frac{1}{2}\right)\right] = 2\left[\frac{2}{3} + \frac{1}{6}\right] = 2\left(\frac{5}{6}\right) = \frac{5}{3}$

$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) dx$

$= 2\int_1^2 x^2(x-1)dx = 2\int_1^2 (x^3 - x^2)dx$

$= 2\left[\frac{x^4}{4} - \frac{x^3}{3}\right]_1^2 = 2\left[\left(\frac{2^4}{4} - \frac{2^3}{3}\right) - \left(\frac{1^4}{4} - \frac{1^3}{3}\right)\right]_1^2 = \frac{17}{6}$

Variance $Var(X) = E(X^2) - [E(X)]^2$

$= \frac{17}{6} - \left(\frac{5}{3}\right)^2 = \frac{17}{6} - \frac{25}{9} = \frac{51-50}{18} = \frac{1}{18}$

(iv) $f(x) = \begin{cases} \frac{1}{2}e^{-\frac{x}{2}} & \text{for } x>0 \\ 0 & \text{otherwise} \end{cases}$

Since ' X ' is a continuous random variable

Mean $E(X) = \int_{-\infty}^{\infty} xf(x)dx$ $\quad [\int_0^{\infty} x^n e^{-ax}dx = \frac{n!}{a^{n+1}}]$

$= \frac{1}{2}\int_0^{\infty} xe^{-\frac{x}{2}}dx = \frac{1}{2}\frac{1!}{(\frac{1}{2})^{1+1}}$ $\quad [n = 1, a = \frac{1}{2}]$

$= \frac{1}{2}(\frac{1}{\frac{1}{4}}) = \frac{4}{2} = 2$

$E(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x)dx$

$= \frac{1}{2}\int_0^{\infty} x^2 \cdot e^{-x/2}dx$ $\quad [n = 2, a = \frac{1}{2}]$

$= \frac{1}{2}\frac{2!}{(\frac{1}{2})^{2+1}} = \frac{1}{2}(\frac{2}{\frac{1}{8}})$

$= \frac{16}{2} = 8$

Variance $Var(X) = E(X^2) - [E(X)]^2 = 8 - 4 = 4$

2.Two balls are drawn in succession without replacement from an urn containing four red balls and three black balls. Let X be the possible outcomes drawing red balls. Find the probability mass function and mean for X.

Solution:

Number of Red balls = 4; Number of Black balls = 3

Total number of balls = 7

Two balls are drawn without replacement

$n(s) = 7C_2$

X denote number of red balls $\quad X = \{0, 1, 2\}$

$P(X = 0) = P(0R, 2B) = \frac{3C_2}{7C_2} = \frac{\frac{3 \times 2}{2 \times 1}}{\frac{7 \times 6}{2 \times 1}} = \frac{3 \times 2}{7 \times 6} = \frac{1}{7}$

$P(X = 1) = P(1R, 1B) = \frac{4C_1\, 3C_1}{7C_2} = \frac{4 \times 3}{\frac{7 \times 6}{2 \times 1}} = \frac{4 \times 3 \times 2}{7 \times 6} = \frac{4}{7}$

$P(X = 2) = P(2R, 0B) = \frac{4C_2}{7C_2} = \frac{\frac{4 \times 3}{2 \times 1}}{\frac{7 \times 6}{2 \times 1}} = \frac{4 \times 3}{7 \times 6} = \frac{2}{7}$

∴ Probability mass function

x	0	1	2
f(x)	1/7	4/7	2/7

$E(X) = \Sigma xf(x) = 0(\frac{1}{7}) + 1(\frac{4}{7}) + 2(\frac{2}{7}) = \frac{8}{7} = 0 + \frac{4}{7} + \frac{4}{7} = \frac{8}{7}$

Question 3.

If μ and σ^2 are the mean and variance of the discrete random variable X, and $E(X + 3) = 10$ and $E(X + 3)^2 = 116$, find μ and σ^2.

Solution:

Mean $= \mu$, $\quad$ Vaniance $= \sigma^2$

Given $E(X + 3) = 10$ and $E(X + 3)^2 = 116$

$E(X) + 3 = 10$ $\quad E(X^2 + 6X + 9) = 116$

$E(X) = 10 - 3$ $\quad E(X^2) + 6E(X) + 9 = 116$

$E(X) = 7$ $\quad E(X^2) + 6(7) + 9 = 116$

∴ Mean $\mu = E(X) = 7$ $\quad E(X^2) + 51 = 116$

$E(X^2) = 116 - 51 = 65$

Variance $Var(X) = E(X^2) - [E(X)]^2$

$65 - 49 = 16 = \sigma^2$ $\quad$ ∴ $\mu = 7$ and $\sigma^2 = 16$

4. Four fair coins are tossed once. Find the probability mass function, mean and variance for number of heads occurred.

SOLUTION:

n = 4, X - random variable denoting no.of heads

X= { 0, 1, 2, 3, 4 } $\quad n(S)=2^4 = 16$

$P(x= 0) = 4C_0 (\frac{1}{2})^4 = 1.\frac{1}{16} = \frac{1}{16}$

$P(x= 1) = 4C_1 (\frac{1}{2})^4 = 4.\frac{1}{16} = \frac{4}{16}$

$P(x= 2) = 4C_2 (\frac{1}{2})^4 = 6.\frac{1}{16} = \frac{6}{16}$

$P(x= 3) = 4C_3 (\frac{1}{2})^4 = 4.\frac{1}{16} = \frac{4}{16}$

$P(x= 4) = 4C_4 (\frac{1}{2})^4 = 1.\frac{1}{16} = \frac{1}{16}$

Probabaility mass function

X	0	1	2	3	4
P(X=x)	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

$P = \frac{1}{2}$ and $q = \frac{1}{2}$

Mean = $np = 4(\frac{1}{2}) = 2$ & Variance = $npq = 4(\frac{1}{2})(\frac{1}{2}) = 1$

Question 7.

The probability density function of the random variable X is given by $f(x) = \begin{cases} 16xe^{-4x} & \text{for } x > 0 \\ 0 & \text{for } x \le 0 \end{cases}$ Find the mean and vaniance of X

Solution:

Given p.d.f. is $f(x) = \begin{cases} 16xe^{-4x} & \text{for } x>0 \\ 0 & \text{for } x\le 0 \end{cases}$

Mean $E(X) = \int_{-\infty}^{\infty} xf(x)dx$

$= 16\int_0^{\infty} x^2 e^{-4x}dx$ $\quad [\int_0^{\infty} x^n e^{-ax}dx = \frac{n!}{a^{n+1}}]$

$= 16 \times \frac{2!}{4^{2+1}} = 16 \times \frac{2 \times 1}{4^3} = 16 \times \frac{2}{64} = \frac{1}{2}$ $\quad [n = 2 ; a = 4]$

$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx = 16\int_0^{\infty} x^3 e^{-4x}dx$ $\quad [n = 3; a = 4]$

$= 16 \times \frac{3!}{4^{3+1}} = 16 \times \frac{3 \times 2 \times 1}{4^4} = 16 \times \frac{6}{256} = \frac{6}{16} = \frac{3}{8}$

Variance $Var(X) = E(X^2) - [E(X)]^2 = \frac{3}{8} - \frac{1}{4} = \frac{3-2}{8} = \frac{1}{8}$

Question 8.

A lottery with 600 tickets gives one prize of ₹200, four prizes of ₹100, and six prizes of ₹50. If the ticket costs is ₹2, find the expected winning amount of a ticket.

Solution: Given, total number of tickets = 600

One prize of Rs. 200; Four prizes of R. 100

Six prizes of Rs. 50

Let ' X ' be the random variable "denotes the winning amount" and it can take the values 200,100 and 50 .

$p(X= 200) = \frac{1}{600}$; $P(X = 100) = \frac{4}{600}$; $P(x = 50) = \frac{6}{600}$

∴ Probability mass function is

x	200	100	50
f(x)	1/600	4/600	6/600

∴ $E(X) = \Sigma xf(x) = \frac{200}{600} + \frac{400}{600} + \frac{300}{600} = \frac{900}{600} = 1.5$

Expected winning amount = Amount won − Cost of lottery

$= 1.50 - 2.00 = -0.50$

ie., Loss of Rs. 0.50

EXERCISE 11.5: 2.

The probability that Mr. Q hits a target at any trial is $\frac{1}{4}$. Suppose he tries at the target 10 times. Find the probability that he hits the target

(i) exactly 4 times (ii) at least one time.

Solution:

Let ' p ' be the probability of hitting the trial

i.e., $p = \frac{1}{4}$, $\therefore q = 1 - p = 1 - \frac{1}{4} = \frac{3}{4}$

number of trials $= n = 10$

$P(X = x) = nC_x p^x q^{n-x}, x = 0, 1, 2, \ldots\ldots, n$

(i) exactly 4 times is

$$P(X = 4) = 10C_4\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right)^{10-4}$$

$$= 10C_4\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right)^6$$

(ii) atleast one time

$P(X \geq 1) = 1 - P(X < 1)$

$= 1 - P(X = 0)$

$$= 1 - 10C_0\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^{10}$$

$$= 1 - \left(\frac{3}{4}\right)^{10}$$

Question 5.

A retailer purchases a certain kind of electronic device from a manufacturer. The manufacturer indicates that the defective rate of the device is 5%. The inspector of the retailer randomly picks 10 items from a shipment. What is the Probability that there will be

(i) at least one defective item (ii) exactly two defective items.

Solution:

Given $n = 10$

Probability of a defective item $= p = 5\% = \frac{5}{100}$

$\therefore q = 1 - p = 1 - \frac{5}{100} = \frac{95}{100}$

Let ' X ' be the random variable denotes the number of defective items.

$P(X = x) = nC_x p^x q^{n-x}, x = 0, 1, 2, \ldots\ldots, n$

(i) Probability that atleast one defective item will be there

$P(X \geq 1) = 1 - P(X < 1) = 1 - [P(X = 0)]$

$$= 1 - \left[10C_0\left(\frac{5}{100}\right)^0\left(\frac{95}{100}\right)^{10-0}\right] = 1 - \left(\frac{95}{100}\right)^{10} = 1 - (0.95)^{10}$$

(ii) Probability that exactly two defective item will be there

$$P(X = 2) = 10C_2\left(\frac{5}{100}\right)^2\left(\frac{95}{100}\right)^8$$

$$= 10C_2(0.05)^2(0.95)^8$$

Question 8.

If $X \sim B(n, p)$ such that $4P(X = 4) = P(x = 2)$ and $n = 6$. Find the distribution, mean and standard deviation.

Solution:

$n = 6.$

$P(X = x) = nC_x p^x q^{n-x}, x = 0, 1, 2, \ldots\ldots, n$

$4P(X = 4) = P(X = 2)$

$4(6C_4 p^4 q^2) = 6C_2 p^2 q^4 \qquad [\therefore 6C_4 = 6C_2]$

$4p^2 = q^2$

$\Rightarrow 4p^2 = (1 - p)^2$

$4p^2 - p^2 + 2p - 1 = 0$

$3p^2 + 2p - 1 = 0$

$(3p - 1)(p + 1) = 0$

$\therefore p = \frac{1}{3}$; $p = -1$ is not possible.

If $p = \frac{1}{3}$ then $q = 1 - \frac{1}{3} = \frac{2}{3}$

Binomial Distribution is $B\left(6, \frac{1}{3}\right)$

Mean $np = 6 \times \frac{1}{3} = 2$

Standard deviation $= \sqrt{npq} = \sqrt{6 \times \frac{1}{3} \times \frac{2}{3}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

Question 9.

In a binomial distribution consisting of 5 independent trials, the probability of 1 and 2 successes are 0.4096 and 0.2048 respectively. Find the mean and variance of the distribution.

solution:

Number of trials $n = 5$

$P(X = x) = nC_x p^x q^{n-x}, x = 0, 1, 2, \ldots\ldots, n$

Given $P(X = 1) = 0.4096$ and $P(X = 2) = 0.2048$

$P(X = 1) = 0.4096 \Rightarrow 5C_1 p^1 q^4 = 5pq^4 = 0.4096$ ---(1)

$P(X = 2) = 0.2048 \Rightarrow 5C_2 p^2 q^3 = 10p^2 q^3 = 0.2048$ ---(2)

Dividing (1) by (2)

$\Rightarrow \frac{5pq^4}{10p^2q^3} = \frac{0.4096}{0.2048}$

$\frac{q}{2p} = 2 \Rightarrow q = 4p$

$\Rightarrow 1 - p = 4p \qquad [\because q = 1 - p]$

$\Rightarrow 5p = 1 \Rightarrow p = \frac{1}{5}$

& $q = 1 - \frac{1}{5} = \frac{5-1}{5} = \frac{4}{5}$

Mean $= np = 5 \times \frac{1}{5} = 1$

Variance $= npq = 5 \times \frac{1}{5} \times \frac{4}{5} = \frac{4}{5}$

CHAPTER 12 - DISCRETE MATHEMATICS (2 MARKS, 3 MARKS, 5 MARKS)

2 - MARKS

EXERCISE 12.1(1)

Determine whether $*$is a binary operation on the sets given below.

Solution:

(i) $a * b = a.|b|$ on R

Let a, b ∈ R, then |b| ∈ R, a.|b| ∈ R

⇒ $*$ is a binary operator on R

(ii) a∗b = min (a,b) on A = {1,2,3, 4,5}

Let a, b ∈ A

$$a*b = \min(a,b) = \begin{cases} a & \text{if } a \le b \\ b & \text{if } b \le a \end{cases}$$

in either case a∗b ∈ A

⇒ $*$ is a binary operator on A

(iii) $(a*b) = a\sqrt{b}$ is binary on R .

Let a, b ∈ R; $\sqrt{b} \notin$ R

∴ $a\sqrt{b} \notin$ R

⇒ a∗b ∉ R

⇒ ∗ is not a binary operator on R

EXERCISE 12.1 (2).

On Z , define ⊗ by (m⊗n)= $m^n + n^m$: ∀ m, n ∈ Z.

Is ⊗ binary on Z ?

SOLUTION:

Let m,n ∈ Z, m > 0 and n < 0 then $m^n \notin Z$

⇒ (m⊗n)= $m^n + n^m \notin Z$, ∴ ⊗ is not a binary on Z

EXERCISE 12.1 (3).

Let ∗ be defined on R by (a∗b) = a + b + ab − 7 .

Is ∗binary on R ? If so, find $3 * \left(\frac{-7}{15}\right)$

SOLUTION:

Let a, b ∈ R, then ab ∈ R

∴ a + b + ab – 7 ∈ R ⇒ (a∗b) = a + b + ab – 7 ∈ R

⇒ ∗ is a binary operator on R

$$3 * \left(\frac{-7}{15}\right) = 3 + \left(\frac{-7}{15}\right) + 3\left(\frac{-7}{15}\right) - 7 = \frac{45-7-21-105}{15} = \frac{-88}{15}$$

---

4. Let A = {a + $\sqrt{5}$b : a,b ∈ Z} . Check whether the usual multiplication is a binary operation on A .

SOLUTION:

Let a + $\sqrt{5}$b, c + $\sqrt{5}$d ∈ A; a,b,c,d ∈ Z

(a + $\sqrt{5}$b)(c + $\sqrt{5}$d)= ac + $\sqrt{5}$ ad + $\sqrt{5}$ bc + 5bd)

= (ac + 5bd)+ $\sqrt{5}$(ad+bc) ∈ A

[ac, bd, ad, bc ∈ Z and ac + 5bd, ad + bc ∈ Z]

∴ usual multiplication is a binary operator on A.

---

6. Fill in the following table so that the binary operation ∗on A = {a,b,c} is commutative.

∗	a	b	c
a	b		
b	c	b	a
c	a		c

SOLUTION:

a ∗ b = b ∗ a = c

a ∗ c = c ∗ a = a

c ∗ b = b ∗ c = a

∗	a	b	c
a	b	c	a
b	c	b	a
c	a	a	c

---

7. Consider the binary operation ∗ defined on the set A = {a,b,c,d} by the following table:

∗	a	b	c	d
A	a	c	b	d
B	d	a	b	c
C	c	d	a	a
D	d	b	a	c

Is it commutative and associative?

Solution:

∗	a	b	c	d
a	a	c	b	d
b	d	a	b	c
c	c	d	a	a
d	d	b	a	c

Commutative Propertty:

a ∗ b = c & b ∗ a = d

a ∗ b ≠ b ∗ a

Commutative property not satisfied

Associative property : a∗(b ∗ c) = (a ∗ b) ∗ c

L.H.S: a∗(b ∗ c) = a ∗ b = c

R.H.S: (a ∗ b) ∗ c = c ∗ c = a

L.H.S ≠ R.H.S. a∗(b ∗ c) ≠ (a ∗ b) ∗ c

ASSOCIATIVE PROPERTY NOT SATISFIED.

EXERCISE 12.2 (1)

Let p : Jupiter is a planet and q : India is an island be any two simple statements. Give verbal sentence describing each of the following statements.

(i) ¬p (ii) p ∧ ¬q (iii) ¬p ∨q (iv) p→¬q (v) p↔q

Solution:

(i) ¬p : Jupiter is not a planet

(ii) p ∧ ¬q: Jupiter is a planet and India is not a island

(iii) ¬p ∨q: Jupiter is not a planet or India is a land

(iv) p→¬q : If Jupiter is a planet then India is not a island

(v) p↔q : Jupiter is not a planet if and only if India is a island

---

Exercise 12.2(2)

Write each of the following sentences in symbolic form using statement variables p and q .

Solution:

p: 19 is a prime number and

q: All angles of a triangle are equal

(i) 19 is not a prime number and all the angles of a triangle are equal. Ans : ¬p ∧ q

(ii) 19 is a prime number or all the angles of a triangle are not equal. Ans : p v q

(iii) 19 is a prime number and all the angles of a triangle are equal. Ans: p ∧ q

(iv) 19 is not a prime number. Ans : ¬p

Exercise 12.2 q.no (3), (4) objectives

3 - MARKS

EXERCISE 12.1

7. Let $A = \begin{pmatrix} 1010 \\ 0101 \\ 1001 \end{pmatrix}$, $B = \begin{pmatrix} 0101 \\ 1010 \\ 1001 \end{pmatrix}$, $C = \begin{pmatrix} 1101 \\ 0110 \\ 1111 \end{pmatrix}$ be any three boolean matricesof the same type. Find (i) A∨B (ii) A∧B (iii) (A∨B) ∧C (iv) (A∧B) ∨C .

SOLUTION:

$$A \vee B = \begin{pmatrix} 1010 \\ 0101 \\ 1001 \end{pmatrix} \vee \begin{pmatrix} 0101 \\ 1010 \\ 1001 \end{pmatrix} = \begin{pmatrix} 1\vee0 & 0\vee1 & 1\vee0 & 0\vee1 \\ 0\vee1 & 1\vee0 & 0\vee1 & 1\vee0 \\ 1\vee1 & 0\vee0 & 0\vee0 & 1\vee1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

$$A \wedge B = \begin{pmatrix} 1010 \\ 0101 \\ 1001 \end{pmatrix} \wedge \begin{pmatrix} 0101 \\ 1010 \\ 1001 \end{pmatrix}$$

$$= \begin{pmatrix} 1\wedge0 & 0\wedge1 & 1\wedge0 & 0\wedge1 \\ 0\wedge1 & 1\wedge0 & 0\wedge1 & 1\wedge0 \\ 1\wedge1 & 0\wedge0 & 0\wedge0 & 1\wedge1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

$$(A \vee B) \wedge C = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \wedge \begin{pmatrix} 1101 \\ 0110 \\ 1111 \end{pmatrix}$$

$$= \begin{pmatrix} 1\wedge1 & 1\wedge1 & 1\wedge0 & 1\wedge1 \\ 1\wedge0 & 1\wedge1 & 1\wedge1 & 1\wedge0 \\ 1\wedge1 & 0\wedge1 & 0\wedge1 & 1\wedge1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

$$(A \wedge B) \vee C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \vee \begin{pmatrix} 1101 \\ 0110 \\ 1111 \end{pmatrix}$$

$$= \begin{pmatrix} 0\vee1 & 0\vee1 & 0\vee0 & 0\vee1 \\ 0\vee0 & 0\vee1 & 0\vee1 & 0\vee0 \\ 1\vee1 & 0\vee1 & 0\vee1 & 1\vee1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

EXERCISE 12.2

Exercise 12.2(5)(i).

Write the converse, inverse, and contrapositive of each of the following implication.

(i) If x and y are numbers such that x = y , then $x^2 = y^2$

Solution:

(i) Conditional statement: $p \to q$

If x and y are numbers such that x = y , then $x^2 = y^2$

(ii) Converse statement: $q \to p$.

If x and y are numbers such that $x^2 = y^2$ then x = y

(iii) Inverse Statement: $\neg p \to \neg q$

If x and y are numbers such that $x \neq y$, then $x^2 \neq y^2$

(iv) Contrapositive statement: $\neg q \to \neg p$

If x and y are numbers such that $x^2 \neq y^2$ then $x \neq y$

Exercise 12.2(5)(ii).

Write the converse, inverse, and contrapositive of each of the following implication.

(ii) If a quadrilateral is a square then it is a rectangle

Solution:

(i) Conditional statement: $p \to q$

If a quadrilateral is a square then it is a rectangle

(ii) Converse statement: $q \to p$.

If a quadrilateral is a rectangle then it is a square

(iii) Inverse Statement: $\neg p \to \neg q$

If a quadrilateral is not a square then it is not a rectangle

(iv) Contrapositive statement: $\neg q \to \neg p$

If a quadrilateral is a not a rectangle then it is a not a square

---

5. (i) Define an operation ∗ on Q, $a * b = \left(\frac{a+b}{2}\right)$; $a, b \in Q$. Examine the closure, commutative, and associative properties satisfied by ∗on ℚ.

(ii) Define an operation ∗ on Q, $a * b = \left(\frac{a+b}{2}\right)$; $a, b \in Q$. Examine the existence of identity and the existence of inverse for the operation ∗on ℚ.

Solution:

Closure property: Let a, b ∈ Q then $\frac{a+b}{2} \in \mathbb{Q}$

$\Rightarrow a * b \in \mathbb{Q}$

∴ closure property satisfied

Commutative property:

Let a, b ∈ Q, to verify $a * b = b * a$

L.H.S: $a * b = \frac{a+b}{2}$ & R.HS: $b * a = \frac{b+a}{2} = \frac{a+b}{2}$ = L.H.S

∴ Commutative property satisfied

Associative property

Let a, b, c ∈ Q, to verify $a * (b * c) = (a * b) * c$

L.H.S: $a * (b * c) = a * \left(\frac{b+c}{2}\right) = \frac{a + \frac{b+c}{2}}{2} = \frac{2a+b+c}{4}$

R.H.S: $(a * b) * c = \left(\frac{a+b}{2}\right) * c = \frac{\frac{a+b}{2} + C}{2} = \frac{a+b+2c}{4}$

L.H.S ≠ R.H.S. Associative property not satisfied

(ii) Identity property:

Let e be the identity element such that

$a * e = a \Rightarrow \frac{a+e}{2} = a \Rightarrow a + e = 2a$

$\Rightarrow e = 2a - a = a$ since e = a which is not unique

So identity property not satisfied

Since identity property not satisfied inverse also not satisfied

---

Exercise 12.2 (6)

Construct the truth table for the following statements.

(i) ¬p∧¬q

Solution:

No of simple statements = 2; No. of rows = $2^2 = 4$

p	q	¬p	¬q	¬p∧¬q
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Last column corresponding to ¬p∧¬q

Exercise 12.2 (6):

Construct the truth table for the following statements.

(ii) ¬(p∧¬q)

Solution:

No of simple statements = 2; No. of rows = $2^2 = 4$

p	q	¬q	p ∧ ¬q	¬(p∧¬q)
T	T	F	F	T
T	F	T	T	F
F	T	F	F	T
F	F	T	F	T

Last column corresponding to **¬(p∧¬q)**

Exercise 12.2 (6): Construct the truth table for

(iii) (p ∨q) ∨ ¬q

Solution: No of simple statements = 2; No. of rows = $2^2 = 4$

p	q	¬q	p ∨ q	(p ∨ q) ∨ ¬q
T	T	F	T	T
T	F	T	T	T
F	T	F	T	T
F	F	T	F	T

Last column corresponding to **(p ∨q) ∨ ¬q**

Exercise 12.2 (6):

Construct the truth table for the following statements.

(iv) $(\neg p\to r)\wedge(p\leftrightarrow q)$

Solution:

No of simple statements = 3; No. of rows = $2^3 = 8$

p	q	r	$\neg p$	T→F F $\neg p\to r$	T → F F F ← T F $p\leftrightarrow q$	$(\neg p\to r)\wedge(p\leftrightarrow q)$
T	T	T	F	T	T	T
T	T	F	F	T	T	T
T	F	T	F	T	F	F
T	F	F	F	T	F	F
F	T	T	T	T	F	F
F	T	F	T	F	F	F
F	F	T	T	T	T	T
F	F	F	T	F	T	F

Last column corresponding $(\neg p\to r)\wedge(p\leftrightarrow q)$

Exercise 12.2: (7)

Verify whether the following compound propositions are tautologies or contradictions or contingency:

(i) $(p\wedge q)\wedge\neg(p\vee q)$

Solution:

No of simple statements = 2; No. of rows = $2^2 = 4$

p	q	$p\wedge q$	$p\vee q$	$\neg(p\vee q)$	$(p\wedge q)\wedge\neg(p\vee q)$
T	T	T	T	F	F
T	F	F	T	F	F
F	T	F	T	F	F
F	F	F	F	T	F

Since last column contains ONLY F So it is contradiction

Exercise 12.2: (7)

Verify whether the following compound propositions are tautologies or contradictions or contingency:

(ii) $((p\vee q)\wedge\neg p)\to q$

Solution:

No of simple statements = 2; No. of rows = $2^2 = 4$

p	q	$p\vee q$	$\neg p$	$(p\vee q)\wedge\neg p$	T → F F $((p\vee q)\wedge\neg p)\to q$
T	T	T	F	F	T
T	F	T	F	F	T
F	T	T	T	T	T
F	F	F	T	F	T

Since last column contains ONLY T so it is tautology

Exercise 12.2: (7)

Verify whether the following compound propositions are tautologies or contradictions or contingency:(iii) $(p\to q)\leftrightarrow(\neg p\to q)$

Solution:

No of simple statements = 2; No. of rows = $2^2 = 4$

p	q	T→F F $p\to q$	$\neg p$	T → F F $\neg p\to q$	T → F F F ← T F $(p\to q)\leftrightarrow(\neg p\to q)$
T	T	T	F	T	T
T	F	F	F	T	F
F	T	T	T	T	T
F	F	T	T	F	F

Since last column contains both T and F it is contigencey

Exercise 12.2: (7)

Verify whether the following compound propositions are tautologies or contradictions or contingency:

(iv) $((p\to q)\wedge(q\to r))\to(p\to r)$

Solution:

No of simple statements = 3; No. of rows = $2^3 = 8$

p	q	r	T→F F $p\to q$	T→F F $q\to r$	$(p\to q)\wedge(q\to r)$	T→F F $p\to r$	$((p\to q)\wedge(q\to r))\to(p\to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Since last column contains ONLY T so it is tautology

Exercise 12.2: (8): Show that (i) $\neg(p\wedge q)\equiv\neg p\vee\neg q$

Solution: No of simple statements = 2; No. of rows = $2^2 = 4$

p	q	$p\wedge q$	L.H.S $\neg(p\wedge q)$	$\neg p$	$\neg q$	R.H.S $\neg p\vee\neg q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

Since column corresponding to L.H.S and R.H.S are identical,
Hence $\neg(p\wedge q)\equiv\neg p\vee\neg q$

Exercise 12.2: (8): Show that (ii) $\neg(p\to q)\equiv p\wedge\neg q$

Solution: No of simple statements = 2; No. of rows = $2^2 = 4$

p	q	T→F F $p\to q$	L.H.S $\neg(p\to q)$	$\neg q$	R.H.S $p\wedge\neg q$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	T	F	F	F
F	F	T	F	T	F

Since column corresponding to L.H.S and R.H.S are identical,
Hence $\neg(p\to q)\equiv p\wedge\neg q$

Exercise 12.2: (9): Prove that $q\to p\equiv\neg p\to\neg q$

Solution: No of simple statements = 2; No. of rows = $2^2 = 4$

p	q	L.H.S T → F F $q\to p$	$\neg p$	$\neg q$	R.H.S T → F F $\neg p\to\neg q$
T	T	T	F	F	T
T	F	T	F	T	T
F	T	F	T	F	F
F	F	T	T	T	T

Since column corresponding to L.H.S and R.H.S are identical,
Hence $q\to p\equiv\neg p\to\neg q$

Exercise 12.2: (10):

Show that $p\to q$ and $q\to p$ are not equivalent

Solution:

No of simple statements = 2; No. of rows = $2^2 = 4$

p	q	T→F F $p\to q$	T → F F $q\to p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

Since column corresponding $p\to q$ AND $q\to p$ are NOT identical . $p\to q$ and $q\to p$ are not equivalent

Exercise 12.2: (11):

Show that ¬(p↔q) ≡ p ↔ ¬q

Solution:

No of simple statements = 2; No. of rows = $2^2 = 4$

p	q	p ↔ q	¬(p↔q)	¬q	p ↔ ¬q
T	T	T	F	F	F
T	F	F	T	T	T
F	T	F	T	F	T
F	F	T	F	T	F

Since column corresponding p → q AND q → p are NOT identical

p→q and q→p are not equivalent

Exercise 12.2: (13):

Using truth table check whether the statements ¬(p∨q)∨(¬p∧q) and ¬p are logically equivalent.

Solution:

No of simple statements = 2; No. of rows = $2^2 = 4$

p	q	p∨q	¬(p∨q)	¬p	¬p∧q	¬(p∨q)∨(¬p∧q)
T	T	T	F	F	F	F
T	F	T	F	F	F	F
F	T	T	F	T	T	T
F	F	F	T	T	F	T

Since column corresponding ¬(p∨q)∨(¬p∧q) and ¬p are identical

Hence ¬(p∨q)∨(¬p∧q) and ¬p are logically equivalent

5 MARKS

Example 12.9

Verify (i) closure property, (ii) commutative property, (iii) associative property, (iv) existence of identity, and (v) existence of inverse for the operation $+_5$ on Z_5 using table corresponding to addition modulo 5.

SOLUTION: Z_5= { [0], [1], [2], [3], [4] } = { 0, 1, 2, 3, 4 }

$+_5$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

CLOSURE PROPERTY:

All the elements in the table are form the set only

Closure property is verified

Commutative property :

Table is symmetric about main diagonal

Commutative property is verified

Associative property:

$+_5$ is alwys associative, Associative property is verified

Identity property:

0 ∈ Z_5 is the identity element, identity property is verified.

Inverse property:

ELEMENT	0	1	2	3	4
INVERSE	0	4	3	2	1

Inverse property is verified

Example 12.10

Verify (i) closure property, (ii) commutative property, (iii) associative property, (iv) existence of identity, and (v) existence of inverse for the operation x_{11} on a subset A = {1,3,4,5,9} of the set of remainders {0,1,2,3,4,5,6,7,8,9,10}.

SOLUTION:

A= {1,3,4,5,9}

x_{11}	1	3	4	5	9
1	1	3	4	5	9
3	3	9	1	4	5
4	4	1	5	9	3
5	5	4	9	3	1
9	9	5	3	1	4

CLOSURE PROPERTY:

All the elements in the table are form the set only

Closure property is verified

Commutative property :

Table is symmetric about main diagonal

Commutative property is verified

Associative property:

x_{11} is alwys associative, Associative property is verified

Identity property:

1 ∈ A is the identity element, identity property is verified.

Inverse property:

ELEMENT	1	3	4	5	9
INVERSE	1	4	3	9	5

Inverse Property satisfied

10. (i) Let A be Q\{1}. Define $*$on A by $\mathbf{x * y = x + y - xy}$. Is $*$ binary on A ? If so, examine the commutative and associative properties satisfied by $*$on A .

(ii) Let A be Q\{1}. Define $*$on A by $\mathbf{x * y = x + y - xy}$. Is $*$binary on A ? If so, examine the existence of identity, existence of inverse properties for the operation $*$on A .

SOLUTION:

Given $\mathbf{A = Q\backslash\{1\}}$ & $\mathbf{x * y = x + y - xy}$

Closure property : Let a,b ∈ A then a ≠1 and b ≠1

Then a + b, ab ∈ A also a + b - ab ∈ A ⇒ a$*$b ∈ A

To verify closure property we must prove a $*$ b ≠1

Let a $*$ b = 1 ⇒ a + b - ab = 1 ⇒ b (1 - a) = 1 - a

⇒ b = $\frac{1-a}{1-a}$ = 1 which is a contradiction b ≠1

a$*$b = a + b - ab ∈ A Closure property verified

Commutative property :

Let a, b ∈ A. To verify a$*$b = b $*$a

L.H.S. : a$*$b = a + b - ab & R.H.S.: b $*$a = b + a - ba

L.H.S = R.H.S Commutative property satisfied.

Associative Property:

Let a, b, c ∈ A. To verify a$*$(b $*$ c) = (a $*$ b) $*$ c

L.H.S.:

a$*$(b $*$ c) = a $*$ (b + c - bc) = a +(b + c - bc)- a(b + c - bc)

= a + b + c - ab - bc - ac + abc

R.H.S:

(a $*$ b) $*$ c = (a + b - ab)$*$c = a + b - ab + c - (a + b - ab)c

= a + b + c - ab - bc- ac + abc

L.H.S = R.H.S Associative property satisfied

(iii) Identity property:

Let e be the identity element such that

a $*$ e = a ⇒ a + e - ae = a ⇒ e(1-a) = a - a = 0

⇒ e = $\frac{0}{1-a}$ = 0∈ A

Identity property satisified

(v) Inverse property:

Let a ∈ A and let a’ ∈ A be the inverse of a such that

a $*$ a’ = o ⇒ a + a’ - aa’ = 0 ⇒ a’ - aa’ = -a ⇒a’ (1-a) = -a

⇒ a’ = $\frac{-a}{1-a}$ ∈ A inverse property satisfied

Example 12.19: Using the equivalence property

Show that p↔q ≡ (p ∧ q) v (¬p ∧ ¬q)

Solution: p↔q ≡ (p → q) ∧ (q → p)

≡ (¬p v q) ∧ (¬q v p)

≡ [(¬p v q) ∧ ¬q] v [(¬p v q) ∧ p]

≡ [¬q ∧ (¬p v q)] v [p ∧ (¬p v q)]

≡ [(¬q ∧ ¬p) v (¬q ∧ q)] v [(p ∧ ¬p) v (p ∧ q)]

≡ [(¬q ∧ ¬p) v 𝔽] v [𝔽 v (p ∧ q)]

≡ (¬q ∧ ¬p) v(p ∧ q)

≡ (p ∧ q) v (¬q ∧ ¬p) Hence proved

5 (i) Define an operation $*$ on Q, $\mathbf{a * b = \left(\frac{a+b}{2}\right)}$; $\mathbf{a, b \in Q}$. Examine the closure, commutative, and associative properties satisfied by $*$on ℚ.

(ii) Define an operation $*$ on Q, $\mathbf{a * b = \left(\frac{a+b}{2}\right)}$; $\mathbf{a, b \in Q}$. Examine the existence of identity and the existence of inverse for the operation $*$on ℚ.

Solution:

Closure property: Let a, b ∈ Q then $\frac{a+b}{2} \in \mathbb{Q}$ ⇒ a $*$ b ∈ ℚ

∴ closure property satisfied

Commutative property:

Let a, b ∈ Q, to verify a $*$ b = b $*$ a

L.H.S: a $*$ b = $\frac{a+b}{2}$ & R.HS: b $*$ a = $\frac{b+a}{2} = \frac{a+b}{2}$ = L.H.S

∴ Commutative property satisfied

Associative property

Let a, b, c ∈ Q, to verify a$*$ (b $*$c) = (a$*$ b) $*$ c

L.H.S: a$*$ (b $*$c) = a $*$ $\left(\frac{b+c}{2}\right) = \frac{a+\frac{b+c}{2}}{2} = \frac{2a+b+c}{4}$

R.H.S: (a$*$ b) $*$ c = ($\frac{a+b}{2}$) $*$ c = $\frac{\frac{a+b}{2}+C}{2} = \frac{a+b+2c}{4}$

L.H.S ≠ R.H.S. Associative property not satisfied

(ii) Identity property:

Let e be the identity element such that

a $*$ e = a ⇒ $\frac{a+e}{2}$ = a ⇒ a + e = 2a

⇒ e = 2a - a = a since e = a which is not unique

So identity property not satisfied

Since identity property not satisfied inverse also not satisfied

Example 12.7

Verify (i) closure property, (ii) commutative property, (iii) associative property, (iv) existence of identity, and (v) existence of inverse for following operation on the given set. $\mathbf{m * n = m + n - mn}$; $\mathbf{n \in Z}$

solution

Closure property : Let a,b ∈ Z

Then a + b, ab ∈ Z also a + b - ab ∈ Z ⇒ a∗b ∈ Z

Closure property verified

Commutative property :

Let a, b ∈ Z. To verify a∗b= b ∗a

L.H.S. : a∗b = a + b - ab & R.H.S.: b ∗a = b + a - ba

L.H.S = R.H.S Commutative property satisfied.

Associative Property:

Let a, b, c ∈ Z. To verify a∗(b ∗ c) = (a ∗ b) ∗ c

L.H.S.:

a∗(b ∗ c) = a ∗ (b + c - bc)

= a +(b + c - bc)- a(b + c - bc)

= a + b + c - ab - bc - ac + abc

R. H.S:

a ∗ b) ∗ c = (a + b - ab)∗c = a + b - ab + c - (a + b - ab)c

= a + b + c - ab - bc- ac + abc

(iv) Identity property:Let e be the identity element such that

a ∗ e = a ⇒ a + e - ae = a ⇒ e(1-a) = a - a = 0

⇒ e = $\frac{0}{1-a}$ = 0∈ Z; Identity property verified

(v) Inverse property:

Let a ∈ Z and let a' ∈ Z be the inverse of a such that

a ∗ a' = o ⇒ a + a' - aa' = 0 ⇒ a' - aa' = -a ⇒a' (1-a) = -a

⇒ a' = $\frac{-a}{1-a}$ ∉ Z inverse property not verified

Exercise 12.1

9.(i) Let $\mathbf{M} = \left\{\begin{pmatrix} \mathbf{x} & \mathbf{x} \\ \mathbf{x} & \mathbf{x} \end{pmatrix} : \mathbf{x} \in \mathbf{R} - \{\mathbf{0}\}\right\}$ and Let * be the matrix multiplication. Determine whether M is closed under *. If so, examine the commutative and associative properties satisfied by * on M.

(ii) Let $\mathbf{M} = \left\{\begin{pmatrix} \mathbf{x} & \mathbf{x} \\ \mathbf{x} & \mathbf{x} \end{pmatrix} : \mathbf{x} \in \mathbf{R} - \{\mathbf{0}\}\right\}$ and let * be the matrix multiplication. Determine whether M is closed under *. If so, examine the existence of an identity, the existence of inverse properties for the operation * on M.

Solution:

$M = \left\{\begin{pmatrix} x & x \\ x & x \end{pmatrix} : x \in R - \{0\}\right\}$; ∗ be the matrix multiplication

Closure property :

Let $A = \begin{pmatrix} a & a \\ a & a \end{pmatrix}$ and a ≠ 0; Let $B = \begin{pmatrix} b & b \\ b & b \end{pmatrix}$ and b ≠ 0

$A * B = \begin{pmatrix} a & a \\ a & a \end{pmatrix}\begin{pmatrix} b & b \\ b & b \end{pmatrix} = \begin{pmatrix} ab+ab & ab+ab \\ ab+ab & ab+ab \end{pmatrix}$

$= \begin{pmatrix} 2ab & 2ab \\ 2ab & 2ab \end{pmatrix} \in M$ (2ab ≠ 0 as a ≠ 0 and b ≠ 0)

Commutative property:

Let A, B ∈ M To verify A ∗ B = B ∗ A

L.H.S. $A * B = \begin{pmatrix} a & a \\ a & a \end{pmatrix}\begin{pmatrix} b & b \\ b & b \end{pmatrix} = \begin{pmatrix} ab+ab & ab+ab \\ ab+ab & ab+ab \end{pmatrix}$

$= \begin{pmatrix} 2ab & 2ab \\ 2ab & 2ab \end{pmatrix}$

R.H.S. $B * A = \begin{pmatrix} b & b \\ b & b \end{pmatrix}\begin{pmatrix} a & a \\ a & a \end{pmatrix} = \begin{pmatrix} ab+ab & ab+ab \\ ab+ab & ab+ab \end{pmatrix}$

$= \begin{pmatrix} 2ab & 2ab \\ 2ab & 2ab \end{pmatrix}$

L.H.S. = R.H.S. ∴ commutative property satisfied

Associative property:

Matrix multiplication is always associative

∴ Associative property is verified.

Identity property:

Let $E = \begin{pmatrix} e & e \\ e & e \end{pmatrix}$ be the identity element such that A∗E = A

To prove: E ∈ M

$A*E = A \Rightarrow \begin{pmatrix} a & a \\ a & a \end{pmatrix}\begin{pmatrix} e & e \\ e & e \end{pmatrix} = \begin{pmatrix} a & a \\ a & a \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 2ae & 2ae \\ 2ae & 2ae \end{pmatrix} = \begin{pmatrix} a & a \\ a & a \end{pmatrix} \Rightarrow 2ae = a \Rightarrow e = \frac{1}{2} \neq 0$

cha $\therefore E = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \in M$ is the identity element

We can prove E∗A = A

INVERSE PROPERTY:

Let $A' = \begin{pmatrix} a' & a' \\ a' & a' \end{pmatrix}$ be the inverse of A ∈M, such that A∗A' = E

To prove A' ∈ M

$A*A' = E \Rightarrow \begin{pmatrix} a & a \\ a & a \end{pmatrix}\begin{pmatrix} a' & a' \\ a' & a' \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 2aa' & 2aa' \\ 2aa' & 2aa' \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \Rightarrow 2aa' = \frac{1}{2} \Rightarrow a' = \frac{1}{4a} \neq 0$ (a≠0)

$\Rightarrow A' = \begin{pmatrix} \frac{1}{4a} & \frac{1}{4a} \\ \frac{1}{4a} & \frac{1}{4a} \end{pmatrix} \in M$ is the inverse of A

We can prove A'∗ A = E

CHAPTER 1 - MATRICES AND DETERMINANTS
5 MARKS

Example 1.1

If $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$, verify that $A(\text{adj } A)=(\text{adj } A)=|A|I_3$.

Example 1.10

If $A = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$, find x and y such that $A^2 + xA + yI_2 = O_2$. Hence, find A^{-1}.

Example 1.12

If $A=\frac{1}{7}\begin{bmatrix} 6 & -3 & a \\ b & -2 & 6 \\ 2 & c & 3 \end{bmatrix}$ is orthogonal, find a , b ,and c , and hence A^{-1}.

EXERCISE 1.1

3. If $F(\alpha) = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}$, show that $[F(\alpha)]^{-1} = F(-\alpha)$.

4. If $A = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}$, show that $A^2 - 3A - 7I_2 = O_2$. Hence find A^{-1}.

Example 1.19

Show that the matrix $\begin{bmatrix} 3 & 1 & 4 \\ 2 & 0 & -1 \\ 5 & 2 & 1 \end{bmatrix}$ is non-singular and reduce it to the identity matrix by elementary row transformations.

Example 1.21 Find the inverse of $A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$ by Gauss-Jordan method.

EXERCISE 1.2

3. Find the inverse of each of the following by Gauss-Jordan method:

(ii) $\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & -2 & -3 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$

Example 1.23

Solve the following system of equations, using matrix inversion method: $2x_1 + 3x_2 + 3x_3 = 5$, $x_1 - 2x_2 + x_3 = -4$, $3x_1 - x_2 - 2x_3 = 3$.

Example 1.24

If $A = \begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$, find the products AB and BA and hence solve the system of equations $x - y + z = 4$, $x - 2y - 2z = 9$, $2x + y + 3z = 1$.

EXERCISE 1.3

1. Solve the following system of linear equations by matrix inversion method:

(iii) $2x + 3y - z = 9$, $x + y + z = 9$, $3x - y - z = 1$

(iv) $x + y + z - 2 = 0$, $6x - 4y + 5z - 31 = 0$, $5x + 2y + 2z = 13$

2. If $A = \begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$, find the products AB and BA and hence solve the system of equations

$x + y + 2z = 1$, $3x + 2y + z = 7$, $2x + y + 3z = 2$.

4. The prices of three commodities A , B and C are ₹ x , y , and z per units respectively. A person P purchases 4 units of B and sells two units of A and 5 units of C . Person Q purchases 2 units of C and sells 3 units of A and one unit of B . Person R purchases one unit of A and sells 3 unit of B and one unit of C . In the process, P , Q ,and R earn ₹15,000, ₹1,000 and ₹4,000 respectively. Find the prices per unit of A , B ,and C . (Use matrix inversion method to solve the problem.)

Example 1.25 Solve, by Cramer's rule, the system of equations $x_1 - x_2 = 3$, $2x_1 + 3x_2 + 4x_3 = 17$, $x_2 + 2x_3 = 7$

Example 1.26

In a T20 match, Chennai Super Kings needed just 6 runs to win with 1 ball left to go in the last over. The last ball was bowled and the batsman at the crease hit it high up. The ball traversed along a path in a vertical plane and the equation of the path is $y = ax^2 + bx + c$ with respect to a xy - coordinate system in the vertical plane and the ball traversed through the points (10,8) , (20,16) , (30,18) , can you conclude that Chennai Super Kings won the match? Justify your answer. (All distances are measured in metres and the meeting point of the plane of the path with the farthest boundary line is (70,0) .)

EXERCISE 1.4

5. A family of 3 people went out for dinner in a restaurant. The cost of two dosai, three idlies and two vadais is ₹150. The cost of the two dosai, two idlies and four vadais is ₹200. The cost of five dosai, four idlies and two vadais is ₹250. The family has ₹350 in hand and they ate 3 dosai and six idlies and six vadais. Will they be able to manage to pay the bill within the amount they had ?

Example 1.27

Solve the following system of linear equations, by Gaussian elimination method :

$4x + 3y + 6z = 25$, $x + 5y + 7z = 13$, $2x + 9y + z = 1$

Example 1.28

The upward speed $v(t)$ of a rocket at time t is approximated by $v(t) = at^2 + bt + c$, $0 \leq t \leq 100$ where a , b , and c are constants. It has been found that the speed at times

$t = 3$, $t = 6$, and $t = 9$ seconds are respectively, 64, 133, and 208 miles per second respectively. Find the speed at time t =15 seconds. (Use Gaussian elimination method.)

EXERCISE 1.5

1. Solve the following systems of linear equations by Gaussian elimination method:

(i) $2x - 2y + 3z = 2$, $x + 2y - z = 3$, $3x - y + 2z = 1$

(ii) $2x + 4y + 6z = 22$, $3x + 8y + 5z = 27$, $-x + y + 2z = 2$

2. If $ax^2 + bx + c$ is divided by $x + 3$, $x - 5$, and $x - 1$, the remainders are 21 , 61 and 9 respectively.Find a , b , and c .

(Use Gaussian elimination method.)

3. An amount of ₹65,000 is invested in three bonds at the rates of 6%, 8% and 10% per annum respectively. The total annual income is ₹4,800. The income from the third bond is ₹600 more than that from the second bond. Determine the price of each bond. (Use Gaussian elimination method.)

4. A boy is walking along the path $y = ax^2 + bx + c$ through the points $(-6, 8)$, $(-2, -12)$, and $(3, 8)$. He wants to meet his friend at P(7,60). Will he meet his friend?

(Use Gaussian elimination method.)

Example 1.29

Test for consistency of the following system of linear equations and if possible solve : $x + 2y - z = 3$,

$3x - y + 2z = 1$, $x - 2y + 3z = 3$, $x - y + z + 1 = 0$.

Example 1.30

Test for consistency of the following system of linear equations and if possible solve:

$4x - 2y + 6z = 8$, $x + y - 3z = -1$, $15x - 3y + 9z = 21$.

Example 1.31

Test for consistency of the following system of linear equations and if possible solve:

$x - y + z = -9$, $2x - 2y + 2z = -18$, $3x - 3y + 3z + 27 = 0$

Example 1.32

Test the consistency of the following system of linear equations

$x - y + z = -9$, $2x - y + z = 4$,

$3x - y + z = 6$, $4x - y + 2z = 7$

Example 1.33

Find the condition on a , b ,and c so that the following system of linear equations has one parameter family of solutions: $x + y + z = a$, $x + 2y + 3z = b$, $3x + 5y + 7z = c$.

Example 1.34

Investigate for what values of λ and μ the system of linear equations $x + 2y + z = 7$, $x + y + \lambda z = \mu$, $x + 3y - 5z = 5$ has (i) no solution (ii) a unique solution (iii) an infinite number of solutions.

EXERCISE 1.6

1. Test for consistency and if possible, solve the following systems of equations by rank method.

(i) $x - y + 2z = 2$, $2x + y + 4z = 7$, $4x - y + z = 4$

(ii) $3x + y + z = 2$, $x - 3y + 2z = 1$, $7x - y + 4z = 5$

(iv) $2x - y + z = 2$, $6x - 3y + 3z = 6$, $4x - 2y + 2z = 4$

2. Find the value of k for which the equations

$kx - 2y + z = 1$, $x - 2ky + z = -2$, $x - 2y + kz = 1$

have (i) no solution (ii) unique solution

(iii) infinitely many solution

3. Investigate the values of λ and μ the system of linear equations $2x + 3y + 5z = 9$, $7x + 3y - 5z = 8$,

$2x + 3y + \lambda z = \mu$, have (I)no solution (ii) a unique solution (iii) an infinite number of solutions.

Example 1.36

Solve the system: $x + 3y - 2z = 0$, $2x - y + 4z = 0$,

$x - 11y + 14z = 0$.

Example 1.37

Solve the system: $x + y - 2z = 0$, $2x - 3y + z = 0$,

$3x - 7y + 10z = 0$, $6x - 9y + 10z = 0$.

Example 1.38

Determine the values of λ for which the following system of equation $(3\lambda - 8)x + 3y + 3z = 0$, $3x + (3\lambda - 8)y + 3z = 0$

$3x + 3y + (3\lambda - 8)z = 0$ has a non-trivial solution

Example 1.39

By using Gaussian elimination method, balance the chemical reaction equation: $C_5H_8 + O_2 \rightarrow CO_2 + H_2O$

Example 1.40

If the system of equations

$px + by + cz = 0$, $ax + qy + cz = 0$, $ax + by + rz = 0$

has a non-trivial solution and $p \neq a$, $q \neq b$, $r \neq c$,

prove that $\frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c} = 2$

CHAPTER 4 - INVERSE TROGNOMETRIC EQUATION

EXAMPLE 4.4 :

Find the domain of $\sin^{-1}(2-3x^2)$

Exercise 4.1 - 6(i) :

Find the domain of $f(x) = \sin^{-1}\left(\frac{x^2+1}{2x}\right)$

Example 4.7

Find the domain of $\cos^{-1}\left(\frac{2+\sin x}{3}\right)$.

Exercise 4.2 - 6(i)

Find the domain of $f(x) = \sin^{-1}\left(\frac{|x|-2}{3}\right) + \cos^{-1}\left(\frac{1-|x|}{4}\right)$

Exercise 4.3-4(ii)

Find the value of $\sin\left(\tan^{-1}\left(\frac{1}{2}\right) - \cos^{-1}\left(\frac{4}{5}\right)\right)$

Exercise 4.3-4(iii)

Find the value of $\cos\left(\sin^{-1}\left(\frac{4}{5}\right) - \tan^{-1}\left(\frac{3}{4}\right)\right)$

Example 4.20

Evaluate $\sin\left[\sin^{-1}\left(\frac{3}{5}\right) + \sec^{-1}\left(\frac{5}{4}\right)\right]$

Example 4.22

If $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$ and $0 < x, y, z < 1$, show that $x^2 + y^2 + z^2 + 2xyz = 1$

Example 4.23

If $a_1, a_2, a_3, \ldots a_n$ is an arithmetic progression with common difference d, prove that

$$\tan\left[\tan^{-1}\left(\frac{d}{1+a_1a_2}\right) + \tan^{-1}\left(\frac{d}{1+a_2a_3}\right) + \ldots + \tan^{-1}\left(\frac{d}{1+a_na_{n-1}}\right)\right] = \frac{a_n-a_1}{1+a_1a_n}$$

Example 4.27:

Solve $\tan^{-1}2x + \tan^{-1}3x = \frac{\pi}{4}$, if $6x^2 < 1$

Example 4.28

Solve $\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$

Example 4.29

Solve $\cos\left(\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)\right) = \sin\left\{\cot^{-1}\left(\frac{3}{4}\right)\right\}$

Exercise 4.5 3(ii)

Find the value of $\cot\left(\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{4}{5}\right)$

Exercise 4.5 3(iii)

Find the value of $\tan\left(\sin^{-1}\frac{3}{5} + \cot^{-1}\frac{3}{2}\right)$.

CHAPTER 10 - APPLICATION PROBLEMS

Example 10.27

The growth of a population is proportional to the number present. If the population of a colony doubles in 50 years, in how many years will the population become triple?

Example 10.28

A radioactive isotope has an initial mass 200mg , which two years later is 50mg . Find the expression for the amount of the isotope remaining at any time. What is its half-life? (half-life mean the time taken for the radioactivity of a specified isotope to fall to half its original value).

Example 10.29

In a murder investigation, a corpse was found by a detective at exactly 8 p.m. Being alert, the detective also measured the body temperature and found it to be 70°F. Two hours later, the detective measured the body temperature again and found it to be 60°F. If the room temperature is 50°F, and assuming that the body temperature of the person before death was 98.6°F, at what time did the murder occur?

[log(2.43)=0.88789 ; log(0.5)=-0.69315]

Example 10.30

A tank contains 1000 litres of water in which 100 grams of salt is dissolved. Brine (Brine is a high-concentration solution of salt (usually sodium chloride) in water) runs in a rate of 10 litres per minute, and each litre contains 5 grams of dissolved salt. The mixture of the tank is kept uniform by stirring. Brine runs out at 10 litres per minute. Find the amount of salt at any time t .

EXERCISE 10.8

1. The rate of increase in the number of bacteria in a certain bacteria culture is proportional to the number present.

Given that the number triples in 5 hours, find how many bacteria will be present after 10 hours?

2. Find the population of a city at any time t, given that the rate of increase of population is proportional to the population at that instant and that in a period of 40 years the population increased from 3,00,000 to 4,00,000.

3. The equation of electromotive force for an electric circuit containing resistance and self-inductance is $\mathbf{E = Ri + L\frac{di}{dt}}$, where E is the electromotive force is given to the circuit, R the resistance and L, the coefficient of induction. Find the current i at time t when $E = 0$.

4. The engine of a motor boat moving at 10 m / s is shut off. Given that the retardation at any subsequent time (after shutting off the engine) equal to the velocity at that time. Find the velocity after 2 seconds of switching off the engine.

5. Suppose a person deposits ₹10,000 in a bank account at the rate of 5% per annum compounded continuously. How much money will be in his bank account 18 months later?

6. Assume that the rate at which radioactive nuclei decay is proportional to the number of such nuclei that are present in a given sample. In a certain sample 10% of the original number of radioactive nuclei have undergone disintegration in a period of 100 years. What percentage of the original radioactive nuclei will remain after 1000 years?

7. Water at temperature 100℃ cools in 10 minutes to 80℃ in a room temperature of 25℃ . Find (i) The temperature of water after 20 minutes (ii) The time when the temperature is 40℃

$$\left[\mathbf{\log_e \frac{11}{15} = -0.3101 \ ; \ \log_e 5 = 1.6094}\right]$$

8. At 10.00 A.M. a woman took a cup of hot instant coffee from her microwave oven and placed it on a nearby Kitchen counter to cool . At this instant the temperature of the coffee was 180°F, and 10 minutes later it was 160°F . Assume that constant temperature of the kitchen was 70°F .

(i)What was the temperature of the coffee at 10.15A.M.?

(ii)The woman likes to drink coffee when its temperature is between 130°F and 140°F between what times should she have drunk the coffee?

9.A pot of boiling water at 100℃ is removed from a stove at time $t = 0$ and left to cool in the kitchen. After 5 minutes, the water temperature has decreased to 80℃ , and another 5 minutes later it has dropped to 65 ℃ . Determine the temperature of the kitchen.

10. A tank initially contains 50 litres of pure water. Starting at time $t = 0$ a brine containing with 2 grams of dissolved salt per litre flows into the tank at the rate of 3 litres per minute. The mixture is kept uniform by stirring and the well-stirred mixture simultaneously flows out of the tank at the same rate. Find the amount of salt present in the tank at any time $t > 0$.

CHAPTER - 1

1. If $|\text{adj}(\text{adj } A)| = |A|^9$, then the order of the square matrix A is **Ans :** 4

2. If A is a 3×3 non-singular matrix such that $AA^T = A^T A$ and $B = A^{-1}A^T$, then $BB^T =$ **Ans :** I_3

3. If $A = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$, $B = \text{adj } A$ and $C = 3A$, then $\frac{|\text{adj } B|}{|C|} =$ **Ans :** $\frac{1}{9}$

4. if $A\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$, then $A =$ **Ans :** $\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$

5. if $A = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$, then $9I_2 - A =$ **Ans :** $2A^{-1}$

6. if $A = \begin{bmatrix} 2 & 0 \\ 1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 \\ 2 & 0 \end{bmatrix}$ then $|\text{adj}(AB)| =$ **Ans :** -80

7. If $P = \begin{bmatrix} 1 & x & 0 \\ 1 & 3 & 0 \\ 2 & 4 & -2 \end{bmatrix}$ is the adjoint of 3×3 matrix A and $|A| = 4$, then x is **Ans :** 11

8. If $A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & -2 & 0 \\ 1 & 2 & -1 \end{bmatrix}$, and $A^{-1} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then the value of a_{23} is **Ans :** -1

9. If A, B and C are invertible matrices of some order, then which one of the following is not true? **Ans :** $\text{adj}(AB) = (\text{adj } A)(\text{adj } B)$

10. If $(AB)^{-1} = \begin{bmatrix} 12 & -17 \\ -19 & 27 \end{bmatrix}$, and $A^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$, then $B^{-1} =$ **Ans :** $\begin{bmatrix} 2 & -5 \\ -3 & 8 \end{bmatrix}$

11. If A^TA^{-1} is symmetric, then $A^2 =$ **Ans:** $(A^T)^2$

12. If A is a non-singular matrix such that $A^{-1} = \begin{bmatrix} 5 & 3 \\ -2 & -1 \end{bmatrix}$, then $(A^T)^{-1} =$ **Ans :** $\begin{bmatrix} 5 & -2 \\ 3 & -1 \end{bmatrix}$

13. if $A = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ x & \frac{3}{5} \end{bmatrix}$, and $A^T = A^{-1}$, then the value of x is **Ans :** $\frac{-4}{5}$

14. if $A = \begin{bmatrix} 1 & \tan\frac{\theta}{2} \\ -\tan\frac{\theta}{2} & 1 \end{bmatrix}$, and $AB = I_2$, then B= **Ans :** $\left(\cos^2\frac{\theta}{2}\right) A^T$

15. if $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, and $A(\text{adj } A) = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$, then $k =$ **Ans :** 1

16. If $A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$, be such that $\lambda A^{-1} = A$, then λ is **Ans :** 19

17. If $\text{adj } A = \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}$, $\text{adj } B = \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}$, then $\text{adj}(AB)$ is **Ans :** $\begin{bmatrix} -6 & 5 \\ -2 & -10 \end{bmatrix}$

18. The rank of the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ -1 & -2 & -3 & -4 \end{bmatrix}$ is **Ans :** 1

19. If $x^a y^b = e^m$, $x^c y^d = e^n$, $\Delta_1 = \begin{vmatrix} m & b \\ n & d \end{vmatrix}$, $\Delta_2 = \begin{vmatrix} a & m \\ c & n \end{vmatrix}$, $\Delta_3 = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ then the values of x and y are respectively,

Ans : $e^{(\Delta 1/\Delta 3)}$, $e^{(\Delta 2/\Delta 3)}$

20. Which of the following is/are correct?

Ans : (i) Adjoint of a symmetric matrix is also a symmetric matrix.

(ii) Adjoint of a diagonal matrix is also a diagonal matrix.

(iii) $A(\text{adj}A) = (\text{adj}A)A = |A| I$

21. If $\rho(A) = \rho([A \mid B])$, then the system $AX = B$ of linear equations is **Ans :** consistent

22. If $0 \leq \theta \leq \pi$ and the system of equations $x + (\sin\theta) y - (\cos\theta)z = 0$, $(\cos\theta)x - y + z = 0$, $(\sin\theta)x + y - z = 0$ has a non- trivial solution then θ is **Ans :** $\frac{\pi}{4}$

23. The augmented matrix of a system of linear equations is $\begin{bmatrix} 1 & 2 & 7 & 3 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & \lambda - 7 & \mu + 5 \end{bmatrix}$.

The system has infinitely many solutions if **Ans :** $\lambda = 7, \mu =$ -5

24. let $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$, and $4B = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & x \\ -1 & 1 & 3 \end{bmatrix}$, If B is the inverse of A , then the value of x is **Ans :** 1

25. If $A=\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, then adj(adj A) is **Ans :** $\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$

CHAPTER - 2

1. $i^n + i^{n+1} + i^{n+2} + i^{n+3}$ is **Ans :** 0

2. The value of $\sum_{i=1}^{13} (i^n + i^{n-1})$ is **Ans :** 1+ i

3. The area of the triangle formed by the complex numbers z, iz, and z + iz in the Argand's diagram is **Ans :** $\frac{1}{2}|z|^2$

4. The conjugate of a complex number is $\frac{1}{i-2}$ Then, the complex number is **Ans :** $\frac{-1}{i+2}$

5. If $z = \frac{(\sqrt{3}+i)^3 (3i+4)^2}{(8+6i)^2}$, then |z| is equal to **Ans :** 2

6. If z is a non zero complex number, such that $2iz^2 = \bar{z}$ then |z| is **Ans :** ½

7. If $|z - 2 + i| \leq 2$, then the greatest value of |z| is **Ans :** $\sqrt{5} + 2$

8. If $\left|z - \frac{3}{z}\right| = 2$, then the least value of |z| is **Ans :** 1

9. If $|z| = 1$, then the value of $\frac{1+z}{1+\bar{z}}$ is **Ans :** z

10. The solution of the equation $|z| - z = 1 + 2i$ is **Ans :** $\frac{3}{2} - 2i$

11. If $|z_1| = 1$, $|z_2| = 2$, $|z_3| = 3$ and $|9z_1 z_2 + 4z_1 z_3 + z_2 z_3| = 12$, then the value of $|z_1 + z_2 + z_3|$ is **Ans :** 2

12. If z is a complex number such that $z \in C$ ¥ R and $z + \frac{1}{z} \in R$, then |z| is **Ans :** 1

13. z_1, z_2, and z_3 are complex numbers such that $z_1 + z_2 + z_3 = 0$ and $|z_1| = |z_2| = |z_3| = 1$ then $z_1^2 + z_2^2 + z_3^2$ is

Ans : 0

14. If $\frac{z-1}{z+1}$, is purely imaginary, then |z| is **Ans :** 1

15. If $z = x + iy$ is a complex number such that $|z + 2| = |z - 2|$, then the locus of z is **Ans :** imaginary axis

16. The principal argument of $\frac{3}{-1+i}$ is **Ans :** $\frac{-3\pi}{4}$

17. The principal argument of $(\sin 40° + i \cos 40°)^5$ is **Ans :** $-110°$

18. If (1+ i) (1+ 2i) (1+ 3i).....(1+ ni) = x + iy , then $2 \cdot 5 \cdot 10 \cdots (1 + n^2)$ is **Ans :** $x^2 + y^2$

19. If $\omega \neq 1$ is a cubic root of unity and $(1+\omega)^7 = A + B\omega$, then (A, B) equals **Ans :** (1, 1)

20. The principal argument of the complex number $\frac{(1+i\sqrt{3})^2}{4i(1-i\sqrt{3})}$ is **Ans :** $\frac{\pi}{2}$

21. If α and β are the roots of $x^2 + x + 1 = 0$, then $\alpha^{2020} + \beta^{2020}$ is **Ans :** -1

22. The product of all four values of $\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^{\frac{3}{4}}$ is **Ans :** 1

23. If $\omega \neq 1$ is a cubic root of unity and $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -\omega^2 - 1 & \omega^2 \\ 1 & \omega^2 & \omega^7 \end{vmatrix} = 3k$ then k is equal to **Ans :** $-\sqrt{3}i$

24. The value of $\left(\frac{1+\sqrt{3}i}{1-\sqrt{3}i}\right)^{10}$ is **Ans :** cis $\frac{2\pi}{3}$

25. If $\omega = \text{cis}\ \frac{2\pi}{3}$, then the number of distinct roots of $\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$ **Ans:** 1

CHAPTER - 3

1. A zero of $x^3 + 64$ is **Ans :** -4

2. If f and g are polynomials of degrees m and n respectively, and if $h(x) = (f \circ g)(x)$, then the degree of h is **Ans :** mn

3. A polynomial equation in x of degree n always has **Ans :** n imaginary roots

4. If α, β, and γ are the zeros of $x^3 + px^2 + qx + r$, then $\sum\frac{1}{\alpha}$ is **Ans :** $-\frac{q}{r}$

5. According to the rational root theorem, which number is not possible rational zero of $4x^7 + 2x^4 - 10x^3 - 5$? **Ans :** $\frac{4}{5}$

6. The polynomial $x^3 - kx^2 + 9x$ has three real zeros if and only if, k satisfies **Ans :** $|k| \geq 6$

7. The number of real numbers in $[0, 2\pi]$ satisfying $\sin^4 x - 2\sin^2 x + 1$ is **Ans :** 2

8. If $x^3 + 12x^2 + 10ax + 1999$ definitely has a positive zero, if and only if **Ans :** $a < 0$

9. The polynomial $x^3 + 2x + 3$ has **Ans :** one negative and two imaginary zeros

10. The number of positive zeros of the polynomial $\sum_{j=0}^{n} {}^{n}C_r(-1)^r x^r$ is **Ans :** n

CHAPTER - 4

1. The value of $\sin^{-1}(\cos x)$, $0 \leq x \leq \pi$ is **Ans :** $\frac{\pi}{2} - x$

2. If $\sin^{-1}x + \sin^{-1}y = \frac{2\pi}{3}$; then $\cos^{-1}x + \cos^{-1}y$ is equal to **Ans :** $\frac{\pi}{3}$

3. $\sin^{-1}\frac{3}{5} - \cos^{-1}\frac{12}{13} + \sec^{-1}\frac{5}{3} - \text{cosec}^{-1}\frac{13}{12}$ is equal to **Ans :** 0

4. If $\sin^{-1}x = 2\sin^{-1}\alpha$ has a solution, then **Ans :** $|\alpha| \leq \frac{1}{\sqrt{2}}$

5. $\sin^{-1}(\cos x) = \frac{\pi}{2} - x$ is valid for **Ans :** $0 \leq x \leq \pi$

6. If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \frac{3\pi}{2}$, the value of $x^{2017} + y^{2018} + z^{2019} - \frac{9}{x^{101}+y^{101}+z^{101}}$ is **Ans :** 0

7. If $\cot^{-1}x = \frac{2\pi}{5}$ for some $x \in R$, the value of $\tan^{-1}x$ is **Ans :** $\frac{\pi}{10}$

8. The domain of the function defined by $f(x) = \sin^{-1}\sqrt{x-1}$ is **Ans :** [1,2]

9. If $x = \frac{1}{5}$, the value of $\cos(\cos^{-1}x + 2\sin^{-1}x)$ is **Ans :** $-\frac{1}{5}$

10. $\tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{2}{9}\right)$ is equal to **Ans :** $\tan^{-1}\left(\frac{1}{2}\right)$

11. If the function $f(x) = \sin^{-1}(x^2 - 3)$, then x belongs to **Ans :** $[-2, -\sqrt{2}] \cup [\sqrt{2}, 2]$

12. If $\cot^{-1}2$ and $\cot^{-1}3$ are two angles of a triangle, then the third angle is **Ans :** $\frac{3\pi}{4}$

13. $\sin^{-1}\left(\tan\frac{\pi}{4}\right) - \sin^{-1}\left(\sqrt{\frac{3}{x}}\right) = \frac{\pi}{6}$. Then x is a root of the equation **Ans :** $x^2 - x - 12 = 0$

14. $\sin^{-1}(2\cos^2 x - 1) + \cos^{-1}(1 - 2\sin^2 x) =$ **Ans :** $\frac{\pi}{2}$

15. If $\cot^{-1}\left(\sqrt{\sin\alpha}\right) + \tan^{-1}\left(\sqrt{\sin\alpha}\right) = u$, then cos 2u is equal to **Ans :** -1

16. If $|x| \leq 1$, then $2\tan^{-1} x - \sin^{-1}\frac{2x}{1+x^2}$ is equal to **Ans :** 0

17. The equation $\tan^{-1} x - \cot^{-1} x = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ has **Ans :** unique solution

18. If $\sin^{-1} x + \cot^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{2}$, then x is equal to **Ans :** $\frac{1}{\sqrt{5}}$

19. If $\sin^{-1}\frac{x}{5} + \operatorname{cosec}^{-1}\frac{5}{4} = \frac{\pi}{2}$ then the value of x is **Ans :** 3

20. $\sin(\tan^{-1} x)$, $|x| < 1$ is equal to **Ans :** $\frac{x}{\sqrt{1+x^2}}$

CHAPTER - 5

1. The equation of the circle passing through (1, 5) and (4,1) and touching y –axis is
$x^2 + y^2 - 5x - 6y + 9 + \lambda(4x + 3y - 19) = 0$ where λ is equal to **Ans :** $0, -\frac{40}{9}$

2. The eccentricity of the hyperbola whose latus rectum is 8 and conjugate axis is equal to half
the distance between the foci is **Ans :** $2\sqrt{3}$

3. The circle $x^2 + y^2 = 4x + 8y + 5$ intersects the line $3x - 4y = m$ at two distinct points if **Ans :** $-35 < m < 15$

4. The length of the diameter of the circle which touches the x -axis at the point (1,0)
and passes through point (2, 3) . **Ans :** $\frac{10}{3}$

5. The radius of the circle $3x^2 + by^2 + 4bx - 6by + b^2 = 0$ is **Ans :** $\sqrt{10}$

6. The centre of circle inscribed in a square formed by the lines $x^2 - 8x - 12 = 0$ and $y^2 - 14y + 45 = 0$ is **Ans :** (4, 7)

7. The equation of the normal to the circle $x^2 + y^2 - 2x - 2y + 1 = 0$ which is parallel to the
line $2x + 4y = 3$ is **Ans :** $x + 2y = 3$

8. If P(x, y) be any point on $16x^2 + 25y^2 = 400$ with foci $F_1(3, 0)$ and $F_2(-3, 0)$ then $PF_1 + PF_2$ is **Ans :** 10

9. The radius of the circle passing through the point (6, 2) two of whose diameter are
$x + y = 6$ and $x + 2y = 4$ is **Ans :** $2\sqrt{5}$

10. The area of quadrilateral formed with foci of the hyperbolas $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ is **Ans :** $2(a^2 + b^2)$

11. If the normals of the parabola $y^2 = 4x$ drawn at the end points of its latus rectum are
tangents to the circle $(x - 3)^2 + (y + 2)^2 = r^2$, then the value of r^2 is **Ans :** 2

12. If $x + y = k$ is a normal to the parabola $y^2 = 12x$, then the value of k is **Ans :** 9

13. The ellipse $E_1 : \frac{x^2}{9} + \frac{y^2}{4} = 1$ is inscribed in a rectangle R whose sides are parallel to the coordinate axes. Another
ellipse E_2 passing through the point (0, 4) circumscribes the rectangle R . The eccentricity of the ellipse is **Ans :** $\frac{1}{2}$

14. Tangents are drawn to the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ parallel to the straight line $2x - y = 1$.
One of the points of contact of tangents on the hyperbola is **Ans :** $\left(\frac{9}{2\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

15. The equation of the circle passing through the foci of the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ having centre at (0,3) is

Ans : $x^2 + y^2 - 6y - 7 = 0$

16.Let C be the circle with centre at (1,1) and radius = 1 . If T is the circle centered at (0, y)

passing through the origin and touching the circle C externally, then the radius of T is equal to **Ans :** $\frac{1}{4}$

17.Consider an ellipse whose centre is of the origin and its major axis is along x-axis. If its

eccentrcity is $\frac{3}{5}$ and the distance between its foci is 6, then the area of the quadrilateral

inscribed in the ellipse with diagonals as major and minor axis of the ellipse is **Ans :** 40

18. Area of the greatest rectangle inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is **Ans :** 2ab

19. An ellipse has OB as semi minor axes, F and F′ its foci and the angle FBF′ is a right angle.

Then the eccentricity Of the ellipse is **Ans :** $\frac{1}{\sqrt{2}}$

20. The eccentricity of the ellipse $(x - 3)^2 + (y - 4)^2 = \frac{y^2}{9}$ is **Ans :** $\frac{1}{3}$

21. If the two tangents drawn from a point P to the parabola $y^2 = 4x$ are at right angles then the locus of P is

Ans : $x = -1$

22. The circle passing through $(1, -2)$ and touching the axis of x at $(3, 0)$ passing through the point **Ans :** $(5, -2)$

23. The locus of a point whose distance from $(-2, 0)$ is $\frac{2}{3}$ times its distance from the line $x=-\frac{9}{2}$ is **Ans :** an ellipse

24. The values of m for which the line $y = mx + 2\sqrt{5}$ touches the hyperbola $16x^2 - 9y^2 = 144$

are the roots of $x^2 - (a + b)x - 4 = 0$, then the value of $(a + b)$ is **Ans :** 0

25.If the coordinates at one end of a diameter of the circle $x^2 + y^2 - 8x - 4y + c = 0$ are (11, 2) ,

the coordinates of the other end are **Ans :** $(-3, 2)$

CHAPTER - 6

1. If $\vec{a}$ and $\vec{b}$ are parallel vectors, then $[\vec{a}, \vec{c}, \vec{b}]$ is equal to **Ans :** 0

2. If a vector $\vec{\alpha}$ lies in the plane of $\vec{\beta}$ and $\vec{\gamma}$, then **Ans :** $[\vec{\alpha}, \vec{\beta}, \vec{\gamma}] = 0$

3. If $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{c} = \vec{c}\cdot\vec{a} = 0$, then the value of $[\vec{a},\vec{b},\vec{c}]$ is **Ans :** $|\vec{a}|\,|\vec{b}|\,|\vec{c}|$

4. If $\vec{a},\vec{b},\vec{c}$ are three unit vectors such that $\vec{a}$ is perpendicular to $\vec{b}$, and is parallel to $\vec{c}$ then $\vec{a}\times(\vec{b}\times\vec{c})$

is equal to **Ans :** $\vec{b}$

5. If $[\vec{a},\vec{b},\vec{c}] = 1$, then the value of $\frac{\vec{a}\cdot(\vec{b}\times\vec{c})}{(\vec{c}\times\vec{a})\cdot\vec{b}}+\frac{\vec{b}\cdot(\vec{c}\times\vec{a})}{(\vec{a}\times\vec{b})\cdot\vec{c}}+\frac{\vec{c}\cdot(\vec{a}\times\vec{b})}{(\vec{c}\times\vec{b})\cdot\vec{a}}$ is **Ans :** 1

6. The volume of the parallelepiped with its edges represented by the vectors $\hat{i} + \hat{j}, \hat{i} + 2\hat{j}, \hat{i} + \hat{j} + \pi\hat{k}$ is **Ans :** π

7. If $\vec{a}$ and $\vec{b}$ are unit vectors such that $[\vec{a},\vec{b},\vec{a}\times\vec{b}] = \frac{\pi}{4}$, then the angle between $\vec{a}$ and $\vec{b}$ is **Ans :** $\frac{\pi}{6}$

8. If $\vec{a} = \vec{i}+\vec{j}+\vec{k}, \vec{b} = \vec{i}+\vec{j}$ and $(\vec{a}\times\vec{b})\times\vec{c} = \lambda\vec{a}+\mu\vec{b}$, then the value of $\lambda+\mu$ is **Ans :** 0

9. If $\vec{a},\vec{b},\vec{c}$ are non-coplanar, non-zero vectors such that $[\vec{a},\vec{b},\vec{c}] = 3$, then $\{[\vec{a}\times\vec{b}, \vec{b}\times\vec{c}, \vec{c}\times\vec{a}]\}^2$ is equal to **Ans :** 81

10. If $\vec{a},\vec{b},\vec{c}$ are three non-coplanar vectors such that $\vec{a}\times(\vec{b}\times\vec{c}) = \frac{\vec{b}+\vec{c}}{\sqrt{2}}$, then the angle between $\vec{a}$ and $\vec{b}$ is **Ans :** $\frac{3\pi}{4}$

11. If the volume of the parallelepiped with $\vec{a}\times\vec{b}$, $\vec{b}\times\vec{c}$, $\vec{c}\times\vec{a}$ as coterminous edges is 8 cubic units, then the volume of the parallelepiped with $(\vec{a}\times\vec{b})\times(\vec{b}\times\vec{c})$, $(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a})$ and $(\vec{c}\times\vec{a})\times(\vec{a}\times\vec{b})$ as coterminous edges is, <u>Ans :</u> 64 cubic units

12. Consider the vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$ such that $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = 0$. Let P_1 and P_2 be the planes determined by the pairs of vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, $\vec{d}$ respectively. Then the angle between P_1 and P_2 is <u>Ans :</u> 0°

13. If $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\times\vec{c}$, where $\vec{a}$, $\vec{b}$, $\vec{c}$ are any three vectors such that $\vec{b}.\vec{c} \neq 0$ and $\vec{a}.\vec{b} \neq 0$, then $\vec{a}$ and $\vec{c}$ are <u>Ans :</u> parallel

14. If $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} - 5\hat{k}$, $\vec{c} = 3\hat{i} + 5\hat{j} - \hat{k}$, then a vector perpendicular to $\vec{a}$ and lies in the plane containing $\vec{b}$ and $\vec{c}$ is <u>Ans :</u> $-17\hat{i} - 21\hat{j} - 97\hat{k}$

15. The angle between the lines $\frac{x-2}{3} = \frac{y-1}{-2}$, $z = 2$ and $\frac{x-1}{1} = \frac{2y+3}{3} = \frac{z+5}{2}$ is <u>Ans :</u> $\frac{\pi}{2}$

16. If the line $\frac{x-2}{3} = \frac{y-1}{-5} = \frac{z+2}{2}$ lies in the plane $x + 3y - \alpha z + \beta = 0$, then (α , β) is <u>Ans :</u> $(-6, 7)$

17. The angle between the line $\vec{r} = (\hat{i} + 2\hat{j} - 3\hat{k}) + t(2\hat{i} + \hat{j} - 2\hat{k})$ and the plane $\vec{r}.(\hat{i} + \hat{j}) + 4 = 0$ is <u>Ans :</u> 45°

18. The coordinates of the point where the line $\vec{r} = (6\hat{i} - \hat{j} - 3\hat{k}) + t(-\hat{i} + 4\hat{k})$ meets the plane $\vec{r}.(\hat{i} + \hat{j} - \hat{k}) = 3$ are <u>Ans :</u> $(5, -1, 1)$

19. Distance from the origin to the plane $3x - 6y + 2z + 7 = 0$ is <u>Ans :</u> 1

20. The distance between the planes $x + 2y + 3z + 7 = 0$ and $2x + 4y + 6z + 7 = 0$ is <u>Ans :</u> $\frac{\sqrt{7}}{2\sqrt{2}}$

21. If the direction cosines of a line are $\frac{1}{c}$, $\frac{1}{c}$, $\frac{1}{c}$ then <u>Ans :</u> $c = \pm\sqrt{3}$

22. The vector equation points $\vec{r} = (\hat{i} - 2\hat{j} - \hat{k}) + t(6\hat{i} - \hat{k})$ represents a straight line passing through the <u>Ans :</u> $(1, -2, -1)$ and $(1, 4, -2)$

23. If the distance of the point (1,1,1) from the origin is half of its distance from the plane $x + y + z + k = 0$, then the values of k are <u>Ans :</u> $3, -9$

24. If the planes $\vec{r}.(2\hat{i} - \lambda\hat{j} + \hat{k}) = 3$ and $\vec{r}.(4\hat{i} + \hat{j} + \mu\hat{k}) = 5$ are parallel , then the value of λ and μ are <u>Ans :</u> $-\frac{1}{2}$, -2

25. If the length of the perpendicular from the origin to the plane $2x + 3y + \lambda z = 1$, $\lambda > 0$ is $\frac{1}{5}$, then the value of λ is <u>Ans :</u> $2\sqrt{3}$

CHAPTER - 7

1. The volume of a sphere is increasing in volume at the rate of $3\pi\ cm^3$ / sec . The rate of change of its radius when radius is $\frac{1}{2}$ cm **Ans :** 3 cm/s

2. A balloon rises straight up at 10 m/s. An observer is 40 m away from the spot where the balloon left the ground. The rate of change of the balloon's angle of elevation in radian per second when the balloon is 30 metres above the ground. **Ans :** $\frac{4}{25}$ radians/sec

3. The position of a particle moving along a horizontal line of any time t is given by $s(t) = 3t^2 - 2t - 8$. The time at which the particle is at rest is **Ans :** $t = \frac{1}{3}$

4. A stone is thrown up vertically. The height it reaches at time t seconds is given by $x = 80t - 16t^2$. The stone reaches the maximum height in time t seconds is given by **Ans :** 2.5

5. The point on the curve $6y = x^3 + 2$ at which y-coordinate changes 8 times as fast as x-coordinate is **Ans :** (4,11)

6. The abscissa of the point on the curve $f(x) = \sqrt{8 - 2x}$ at which the slope of the tangent is -0.25 ? **Ans :** -4

7. The slope of the line normal to the curve $f(x) = 2\cos 4x$ at $x = \frac{\pi}{12}$ is **Ans :** $\frac{\sqrt{3}}{12}$

8. The tangent to the curve $y^2 - xy + 9 = 0$ is vertical when **Ans :** $y = \pm 3$

9. Angle between $y^2 = x$ and $x^2 = y$ at the origin is **Ans :** $\frac{\pi}{2}$

10. What is the value of the limit $\lim_{x\to\infty}\left(\cot x - \frac{1}{x}\right)$? **Ans :** 0

11. The function $\sin^4 x + \cos^4 x$ is increasing in the interval **Ans :** $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$

12. The number given by the Rolle's theorem for the function $x^3 - 3x^2$, $x \in [0,3]$ is **Ans :** 2

13. The number given by the Mean value theorem for the function $\frac{1}{x}$, $x \in [1,9]$ is **Ans :** 3

14. The minimum value of the function $|3 - x| + 9$ is **Ans :** 9

15. The maximum slope of the tangent to the curve $y = e^x \sin x$, $x \in [0,2\pi]$ is at **Ans :** $x = \frac{\pi}{2}$

16. The maximum value of the function $x^2 e^{-2x}$, $x > 0$ is **Ans :** $\frac{1}{e^2}$

17. One of the closest points on the curve $x^2 - y^2 = 4$ at the point (6,0) is **Ans :** $(3, \sqrt{5})$

18. The maximum product of two positive numbers , when their sum of the squares is 200 is **Ans :** 100

19. The curve $y = ax^4 + bx^2$ with $ab > 0$ **Ans :** has no points of inflection

20. The point of inflection of the curve $y = (x - 1)^3$ is **Ans :** (1,0)

CHAPTER – 8

1. A circular template has a radius of 10 cm . The measurement of radius has an approximate error of 0.02 cm . Then the percentage error in calculating area of this template is **Ans :** 0.4%

2. The percentage error of fifth root of 31 is approximately how many times the percentage error in 31 ? **Ans :** $\frac{1}{5}$

3. If $u(x,y) = e^{x^2+y^2}$, then $\frac{\partial u}{\partial x}$ is equal to **Ans :** 2xu

4. If $v(x,y) = \log(e^x + e^y)$, then $\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}$ is equal to **Ans :** 1

5. If $w(x,y) = x^y$, $x > 0$, then $\frac{\partial w}{\partial x}$ is equal to **Ans :** yx^{y-1}

6. If $f(x,y) = e^{xy}$, then $\frac{\partial^2 f}{\partial x \partial y}$ is equal to **Ans :** $(1 + xy)e^{xy}$

7. If we measure side of a cube to be 4 cm with an error of 0.1 cm,then the error in our calculation of the volume is **Ans :** 4.8 cu.cm

8. The change in the surface area $S = 6x^2$ of a cube when the edge length varies from x_0 to $x_0 + dx$ is **Ans :** $12x_0dx$

9. The approximate change in the volume V of a cube of side x metres caused by increasing the side by 1% is **Ans :** $0.03x^2m^3$

10. If $g(x,y) = 3x^2 - 5y + 2y^2$, $x(t) = e^t$ and $y(t) = \cos t$, then $\frac{dg}{dt}$ is equal to **Ans :** $6e^{2t} + 5\sin t - 4\cos t \sin t$

11. If $f(x) = \frac{x}{x+1}$, then its differential is given by **Ans :** $\frac{1}{(x+1)^2}dx$

12. If $u(x,y) = x^2 + 3xy + y - 2019$, then $\left.\frac{\partial u}{\partial x}\right|_{(4,-5)}$ is equal to **Ans :** -7

13. Linear approximation for $g(x) = \cos x$ at $x = \frac{\pi}{2}$ is **Ans :** $-x + \frac{\pi}{2}$

14. If $w(x,y,z) = x^2(y - z) + y^2(z - x) + z^2(x - y)$, then $\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z}$ is **Ans :** 0

15. If $f(x,y,z) = xy + yz + zx$, then $f_x - f_z$ is equal to **Ans :** $z - x$

CHAPTER - 9

1. The value of $\int_0^{\frac{2}{3}} \frac{dx}{\sqrt{4-9x^2}}$ is **Ans :** $\frac{\pi}{6}$

2. The value of $\int_{-1}^{2} |x|\, dx$ is **Ans :** $\frac{5}{2}$

3. For any value of $n \in Z$, $\int_0^{\pi} e^{\cos^2 x}\cos^3[(2n+1)x]\, dx$ is **Ans :** 0

4. The value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x \cos x\, dx$ is **Ans :** $\frac{2}{3}$

5. The value of $\int_{-4}^{4} \left[\tan^{-1}\left(\frac{x^2}{x^4+1}\right) + \tan^{-1}\left(\frac{x^4+1}{x^2}\right)\right] dx$ is **Ans :** 4π

6. The value of $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\frac{2x^7-3x^5+7x^3-x+1}{\cos^2 x}\right) dx$ is **Ans :** 2

7. If $f(x) = \int_0^x t\cos t\, dt$, then $\frac{df}{dx} =$ **Ans :** $x\cos x$

8. The area between $y^2 = 4x$ and its latus rectum is **Ans :** $\frac{8}{3}$

9. The value of $\int_0^1 x(1-x)^{99}dx$ is **Ans :** $\frac{1}{10100}$

10. The value of $\int_0^1 \frac{dx}{1+5^{\cos x}}$ is **Ans :** $\frac{\pi}{2}$

11. If $\frac{\Gamma(n+2)}{\Gamma(n)} = 90$ then n is **Ans :** 9

12. The value of $\int_0^{\frac{\pi}{6}} \cos^3 3x\, dx$ is **Ans :** $\frac{2}{9}$

13. The value of $\int_0^{\pi} \sin^4 x\, dx$ is **Ans :** $\frac{3\pi}{8}$

14. The value of $\int_0^{\infty} e^{-3x}x^2 dx$ is **Ans :** $\frac{2}{27}$

15. If $\int_0^a \frac{1}{4+x^2}\, dx = \frac{\pi}{8}$ then a is **Ans :** 2

16. The volume of solid of revolution of the region bounded by $y^2 = x(a-x)$ about x-axis is **Ans :** $\frac{\pi a^3}{6}$

17. If $f(x) = \int_1^x \frac{e^{\sin u}}{u}\, du$, $x > 1$ and $\int_1^3 \frac{e^{\sin x^2}}{x} dx = \frac{1}{2}[f(a) - f(1)]$, then one of the possible value of a is **Ans :** 9

18. The value of $\int_0^1 (\sin^{-1} x)^2 dx$ is **Ans :** $\frac{\pi^2}{4} - 2$

19. The value of $\int_0^a \left(\sqrt{a^2 - x^2}\right)^3 dx$ is **Ans :** $\frac{3\pi a^4}{16}$

20. If $\int_0^x f(t)dt = x + \int_x^1 tf(t)dt$, then the value of f(1) is **Ans :** $\frac{1}{2}$

CHAPTER 10

1. The order and degree of the differential equation $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^{1/3} + x^{1/4} = 0$ are respectively **Ans :** 2, 3

2. The differential equation representing the family of curves $y = A\cos(x + B)$,

   where A and B are parameters, is **Ans :** $\frac{d^2y}{dx^2} + y = 0$

3. The order and degree of the differential equation $\sqrt{\sin x}(dx + dy) = \sqrt{\cos x}(dx - dy)$ is **Ans :** 1, 1

4. The order of the differential equation of all circles with centre at (h, k) and radius ‘a’ is **Ans :** 3

5. The differential equation of the family of curves $y = Ae^x + Be^{-x}$, where A and B are arbitrary constants is **Ans :** $\frac{d^2y}{dx^2} - y = 0$

6. The general solution of the differential equation $\frac{dy}{dx} = \frac{y}{x}$ is **Ans :** $y = kx$

7. The solution of the differential equation $2x\frac{dy}{dx}y = 3$ represents **Ans :** parabola

8. The solution of $\frac{dy}{dx} + p(x)y = 0$ is **Ans :** $y = ce^{-\int p dx}$

9. The integrating factor of the differential equation $\frac{dy}{dx} + y = \frac{1+y}{x}$ is **Ans :** $\frac{e^x}{x}$

10. The integrating factor of the differential equation $\frac{dy}{dx} + P(x)y = Q(x)$ is x . then P(x) **Ans :** $\frac{1}{x}$

11. The degree of the differential equation $y(x) = 1 + \frac{dy}{dx} + \frac{1}{1\cdot 2}\left(\frac{dy}{dx}\right)^2 + \frac{1}{1\cdot 2\cdot 3}\left(\frac{dy}{dx}\right)^3 + \ldots$ is **Ans :** 1

12. If p and q are the order and degree of the differential equation $y\frac{dy}{dx} + x^3\left(\frac{d^2y}{dx^2}\right) + xy = \cos x$, when **Ans :** $p > q$

13. The solution of the differential equation $\frac{dy}{dx} + \frac{1}{\sqrt{1-x^2}} = 0$ is **Ans :** $y + \sin^{-1} x = c$

14. The solution of the differential equation $\frac{dy}{dx} = 2xy$ is **Ans :** $y = Ce^{x^2}$

15. The general solution of the differential equation $\log\left(\frac{dy}{dx}\right) = x + y$ is **Ans :** $e^x + e^{-y} = C$

16. The solution of $\frac{dy}{dx} = 2^{y-x}$ is **Ans :** $\frac{1}{2^x} - \frac{1}{2^y} = C$

17. The solution of the differential equation $\frac{dy}{dx} = \frac{y}{x} + \frac{\emptyset\left(\frac{y}{x}\right)}{\emptyset'\left(\frac{y}{x}\right)}$ is **Ans :** $\emptyset\left(\frac{y}{x}\right) = kx$

18. If sin x is the integrating factor of the linear differential equation $\frac{dy}{dx} + Py = Q$, then P is **Ans :** cot x

19. The number of arbitrary constants in the general solutions of order n and n +1are respectively **Ans :** n, n +1

20. The number of arbitrary constants in the particular solution of a differential equation of third order is **Ans :** 0

21. Integrating factor of the differential equation $\frac{dy}{dx} = \frac{x+y+1}{x+1}$ is **Ans :** $\frac{1}{x+1}$

22. The population P in any year t is such that the rate of increase in the population is proportional to the population. Then **Ans :** $P = Ce^{kt}$

23. P is the amount of certain substance left in after time t. If the rate of evaporation of the substance is proportional to the amount remaining, then **Ans :** $P = Ce^{-kt}$

24. If the solution of the differential equation $\frac{dy}{dx} = \frac{ax+3}{2y+f}$ represents a circle, then the value of a is **Ans :** -2

25. The slope at any point of a curve y = f(x) is given by $\frac{dy}{dx} = 3x^2$ and it passes through (-1,1). Then the equation of the curve is **Ans :** $y = x^3 + 2$

CHAPTER 11

1. Let X be random variable with probability density function $f(x) = \begin{cases} \frac{2}{x^3} & x \geq 1 \\ 0 & x < 1 \end{cases}$

Which of the following statement is correct? **Ans :** mean exists but variance does not exist

2. A rod of length 2l is broken into two pieces at random. The probability density function of the shorter of the two pieces is $f(x) = \begin{cases} \frac{1}{l} & 0 < x < l \\ 0 & l \leq x \leq 2l \end{cases}$. The mean and variance of the shorter of the two pieces are respectively **Ans :** $\frac{l}{2}, \frac{l^2}{12}$

3. Consider a game where the player tosses a six-sided fair die. If the face that comes up is 6, the player wins ₹36, otherwise he loses ₹ k^2 , where k is the face that comes up k = {1, 2, 3, 4, 5}. The expected amount to win at this game in ₹ is **Ans :** $-\frac{19}{6}$

4. A pair of dice numbered 1, 2, 3, 4, 5, 6 of a six-sided die and 1, 2, 3, 4 of a four-sided die is rolled and the sum is determined. Let the random variable X denote this sum. Then the number of elements in the inverse image of 7 is **Ans :** 4

5. A random variable X has binomial distribution with n = 25 and p = 0.8 then standard deviation of X is **Ans :** 2

6. Let X represent the difference between the number of heads and the number of tails obtained when a coin is tossed n times. Then the possible values of X are **Ans :** 2i–n, i = 0,1,2...n

7. If the function $f(x) = \frac{1}{12}$ for a<x<b , represents a probability density function of a continuous random variable X, then which of the following cannot be the value of a and b? **Ans :** 16 and 24

8. Four buses carrying 160 students from the same school arrive at a football stadium. The buses carry, respectively, 42, 36, 34, and 48 students. One of the students is randomly selected. Let X denote the number of students that

were on the bus carrying the randomly selected student . One of the 4 bus drivers is also randomly selected. Let Y denote the number of students on that bus. Then E(X) and E(Y) respectively are **Ans :** 40.75, 40

9. Two coins are to be flipped. The first coin will land on heads with probability 0.6, the second with Probability 0.5. Assume that the results of the flips are independent, and let X equal the total number of heads that result. The value of E(X) is **Ans :** 1.1

10. On a multiple-choice exam with 3 possible destructives for each of the 5 questions, the probability that a student will get 4 or more correct answers just by guessing is **Ans :** $\frac{11}{243}$

11. If $P(X = 0) = 1 - P(X = 1)$. If $E(X) = 3Var(X)$, then $P(X = 0)$ is **Ans :** $\frac{1}{3}$

12. If X is a binomial random variable with expected value 6 and variance 2.4, then $P(X = 5)$ is **Ans :** $\binom{10}{5}\left(\frac{3}{5}\right)^5\left(\frac{2}{5}\right)^5$

13. The random variable X has the probability density function

$f(x) = \begin{cases} ax + b & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$ and $E(X) = \frac{7}{12}$, then a and b are respectively **Ans :** 1 and $\frac{1}{2}$

14.Suppose that X takes on one of the values 0, 1, and 2. If for some constant k,

$P(X = i) = k\ P(X = i - 1)$ for $i = 1 , 2$ and $P(X = 0) = \frac{1}{7}$, then the value of k is **Ans :** 2

15. Which of the following is a discrete random variable?

Ans : I. The number of cars crossing a particular signal in a day.

II. The number of customers in a queue to buy train tickets at a moment.

16. If $f(x) = \begin{cases} 2x & 0 \le x \le a \\ 0 & \text{otherwise} \end{cases}$ is a probability density function of a random variable, then the value of a is **Ans :** 1

17. The probability mass function of a random variable is defined as:

x	-2	-1	0	1	2
f(x)	k	2k	3k	4k	5k

. Then E(X) is equal to: **Ans :** $\frac{2}{3}$

18. Let X have a Bernoulli distribution with mean 0.4, then the variance of (2X–3) is **Ans :** 0.96

19. If in 6 trials, X is a binomial variable which follows the relation 9P(X=4) = P(X=2), then the probability of success is **Ans :** 0.25

20. A computer salesperson knows from his past experience that he sells computers to one in every twenty customers who enter the showroom. What is the probability that he will sell a computer to exactly two of the next three customers ? **Ans :** $\frac{57}{20^3}$

CHAPTER – 12

1. A binary operation on a set S is a function from **Ans :** (S×S) →S

2. Subtraction is not a binary operation in **Ans :** N

3. Which one of the following is a binary operation on N ? **Ans :** Multiplication

4. In the set R of real numbers ‘ ∗ ’ is defined as follows. Which one of the following is not a binary operation on R ? **Ans :** $a * b = a^b$

5. The operation ∗ defined by $a * b = \frac{ab}{7}$ is not a binary operation on **Ans :** Z

6. In the set Q define $a \odot b = a+b+ab$. For what value of y, $3 \odot (y \odot 5) = 7$? **Ans :** $y = \frac{-2}{3}$

7. If $a * b = \sqrt{a^2+b^2}$ on the real numbers then $*$ is **Ans :** both commutative and associative

8. Which one of the following statements has the truth value T ? **Ans :** $\sqrt{5}$ is an irrational number

9. Which one of the following statements has truth value F ? **Ans :** Chennai is in China or $\sqrt{2}$ is an integer

10. If a compound statement involves 3 simple statements, then the number of rows in the truth table is **Ans :** 8

11. Which one is the inverse of the statement $(p \vee q) \rightarrow (p \wedge q)$? **Ans :** $(\neg p \wedge \neg q) \rightarrow (\neg p \vee \neg q)$

12. Which one is the contrapositive of the statement $(p \vee q) \rightarrow r$? **Ans :** $\neg r \rightarrow (\neg p \wedge \neg q)$

13. The truth table for $(p \wedge q) \vee \neg q$ is given below

p	q	$(p \wedge q) \vee \neg q$
T	T	(a)
T	F	(b)
F	T	(c)
F	F	(d)

Which one of the following is true?

	(a)	(b)	(c)	(d)
Ans:	T	T	F	T

14. In the last column of the truth table for $\neg(p \vee \neg q)$ the number of final outcomes of the truth value 'F ' are **Ans :** 3

15. Which one of the following is incorrect? For any two propositions p and q , we have **Ans :** $\neg(p \vee q) \equiv \neg p \vee \neg q$

16.

p	q	$(p \wedge q) \rightarrow \neg q$
T	T	(a)
T	F	(b)
F	T	(c)
F	F	(d)

Which one of the following is correct for the truth value of $(p \wedge q) \rightarrow \neg q$?

	(a)	(b)	(c)	(d)
Ans:	F	T	T	T

17. The dual of $\neg(p \vee q) \vee [p \vee (p \wedge \neg r)]$ is **Ans :** $\neg(p \wedge q) \wedge [p \wedge (p \vee \neg r)]$

18. The proposition $p \wedge (\neg p \vee q)$ is **Ans :** ogically equivalent to $p \wedge q$

19. Determine the truth value of each of the following statements:

(a) 4+2=5 and 6+3=9
(b) 3+2=5 and 6+1=7
(c) 4+5=9 and 1+2= 4
(d) 3+2=5 and 4+7=11

	(a)	(b)	(c)	(d)
Ans :	F	T	F	T

20. Which one of the following is not true? **Ans :** If p and q are any two statements then $p \leftrightarrow q$ is a tautology.